# A Transparant Economy: Some Sketches

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July 28, 2024

**Abstract**

An economy is defined as transparent if its participants truthfully report their wants or needs and accurately represent objective reality. The goal of a transparent economy is to combat the environmental collapse and to reduce unfair distribution of goods and labour. To reach this goal, several considerations suggest a moneyless economy, where scarce goods are distributed using weighted gains. This is a simple case of rationing along a fixed path (Moulin 1999, J. Econ. Th. 84, pp. 41-72). The weight is the accounted effort or effort index, that is, the accounted labour duration per maximum possible labour duration, where 'accounted' is defined as follows. People report each others' effort. They estimate the effort in advance and the total estimate is a 'soft bound' above which the effort may be capped to yield the accounted effort. This soft bound discourages overreporting the efforts.

The main question is how to discourage collusion when estimating the effort. (Question 12 on page 14.) For example, two participants report great estimates of effort and register equally large efforts of each other without having worked at all. A threat of inspection of the real effort discourages collusion, but probably insufficiently. Several other (research) questions remain to be answered in order to improve or reject this economy.

This document is only a sketch of design considerations. Four appendices seem to offer novel ideas, but only the last appendix presents a usable result: merging for fixed-path rationing (Appendix A.3), need-based cake cutting with random selection (Appendix E), a planning game (Appendix F), and bankruptcy with claims guarantees (Appendix G) which is used to impose the soft bounds on the effort index.

# Contents

# 1 Introduction

The *fraud age* – that is the term, in the same vein as the stone age and the bronze age, which Paul Lafargue coined to characterise two centuries of capitalism. In his view, fraudulence is omnipresent in capitalism and licensed by the delusional need for competition and labour, which leads to exploitation, waste, and 'false needs.' [23, pp. 11, 47-49]. That was in 1883. We would now call this age the *frauducene*, a word more specific than 'anthropocene,' which refers to human activity as the cause of environmental collapse. Overproduction, overconsumption, competition, and inefficiency are the main drivers of resource depletion, mass extinction, and pollution, such as by carbon dioxide, which leads to global warming of the atmosphere and the oceans [8, 37]. The free market still bears the same traits of fraud as it had in 1883, such as planned obsolescence and deceptive advertising. Additionally, it engenders the delusion that there is little room or need to take responsibility, in particular, for acknowledging reality. So does the hierarchical organisation, where superordinates and subordinates can hold each other responsible. This is not to say that capitalism as such is the culprit. Rather, it illustrates this document's point of departure – the human propensity to use any system to sustain illusions and to deceive.

Therefore, instead of an economic mechanism that encourages such fraud and that conceals reality, we need a *transparant economy*, which is defined as the distribution of scarce goods and arrangement of labour that meets two conditions. First, true needs or wants are elicited, not by a hierarchical organisation (or authority) but by participants themselves because they engage in a mechanism (or instititution) where truth-telling is most profitable, so-called strategy-proofness. The second condition is the accurate representation of objective reality so as to prudently account for it. At this point, some authority may come into play, for example, by limiting resource use. Possible boundary conditions are Pareto efficiency (no goods are wasted) and the profitability of joining the economy, which goes by the technical term 'individual rationality.' A 'classical' exchange economy can not meet these two conditions and strategy-proofness, according to a theorem of Hurwicz [4]. Fairness in some sense is another desideratum. However, many combinations of such conditions are subject to similar impossibility results.

This foretells that a compromise should be struck when designing a transparent economy, which is the purpose of this document. The economy focuses on small (possibly overlapping) groups, so that any 'loop holes' (options to collude or misreport otherwise) are not utilised because of people's propensity to cooperate, especially in small groups on which they depend for a considerable time.

In this document, only design considerations are presented that lead to several variations of a model based on fixed-path rationing [30]: Once the disadvantages of one variation are identified, they are overcome in a next variation, which in turn reveals other disadvantages, and so on. These variations and considerations are presented to 'map the territory,' that is, to allow reconsidering assumptions and to prompt finding yet other variations of the model. A hybrid search revealed no similar model. To identify this economy in advance, it is called *eranism*, after the classical Greek word 'eranos' for generalised reciprocity ('mutual aid') as explained in Appendix C.

This document is organised as follows. Section 2 presents some design considerations for a transparent economy, which lead to fixed-path rationing. Such rationing (or rather: distribution) is the basis for several variations, treated in Section 3. In Section 4, multiple groups are discussed. The conclusion (Section 5) is that no design can result at this stage.

# 2 Design Considerations

Following are the considerations for choosing a particular implementation of the desired small-scale, transparent economy. These considerations are listed to explain why other implementations are inferior.

There are many disadvantages of money [24, pp. 94, 133], which all seem to spring from money's concealment of physical and social realities by its mediation between consumer and producer. Similarly, in centralised, hierarchical organisations, departments and superordinates are intermediate entities that invite to hide realities, more specifically, to eschew responsibilities. The responsibility becomes 'free floating' or 'diffused' such as in the banking business [20, p. 307] but the lack of organisation can equally induce a lack of responsibility or 'free riding'. (Additionally, money and organisations may also 'lead a life of their own,' so-called reification.) As money and centralised organisations conceal realities, any transparent economy best is moneyless and arranged with the least possible centralised management, which may be feasible using mechanism and institution design.

The first goal of a transparent economy is to accurately represent needs or wants. To that end, a mechanism (or institution) should be found with which telling the truth is most profitable for everyone. Such a mechanism is called strategy-proof or incentive-compatible.

A preliminary question is whether needs or wants should be elicited. An amount of goods or services is a *need* if without it, one physically malfunctions, and a *want* if it gratifies psychological desires. The needs-wants dichotomy may be deceptive because the wanted objects also have meanings and therefore are as essential as needs, whereas needs have only increased due to the complexity of society [16, pp. 335-338]. Be that as it may, in the case of great scarcity, need-based mechanisms (triage) may be mandatory.

**Question 1** *Does a mechanism exists for which telling one's needs (instead of wants) is strategy-proof? The needs are private information, to be distinguished from objective needs, as in welfarism [29, p. 166]. Should the need be modelled as a dip instead of a peak? For single-dipped preferences, strategy-proof mechanisms select from a very limited number of choices [6] or (at least for non-probabilistic or infinitely divisible goods) allocate all to a single person [13]. A candidate solution is a two-stage procedure when there are both single-peaked and single-dipped preferences [2].*

*Or should needs be modeled as a threshold for individual rationality, that is, participation? A still unsatisfactory such approach is in Appendix E.*

For now, consider preferences (wants) that are single-peaked, possibly peaking at the maximum available quantity. Such preferences do not lead to well-known impossibility results [7].

The second goal of a transparent economy is to accurately represent any undesirable state of affairs or prospect of the future, a future which is often discounted. A faithful representation of reality or prediction can partially be achieved by creating artificial scarcity to prevent greater scarcity or to reduce pollution, such as by carbon dioxide. This requires a rationing procedure, which is needed anyway to distribute scarce goods in a strategy-proof, efficient, and possibly fair way. (If goods are never scarce, there is less need for an economy.) It seems inevitable that a centralised body determines the amount to be distributed but the exact determination is a complex affair that is beyond the scope of this document. However, the desired rations (requests for a portion of the scarce good) are not determined by an authority but declared by the participants themselves, where strategy-proofness discourages misreporting these requests..

If the distribution of goods (with preferred amounts) is to be efficient and strategy-proof, and if it meets two 'bouundary conditions' (consistency and resource-monotonicity) then it must be fixed-path rationing [14, 30]. For now, restrict it to the method of weighted gains. See Appendix A for details. Also, let the weight in the weighted gains be the effort or effort index, that is, labour duration per maximum labour duration, to be elaborated on shortly.

**Example 1** *To illustrate that it is fair to have the weights equal effort, let three participants A, B, and C have claims* $2$*. Their weights (relative efforts) are* $\frac{1}{6}$*,* $\frac{1}{3}$*, and* $\frac{1}{2}$*. To be distributed is* $5$ *but requested is* $6$*. The allotments are* $1$*,* $2$*, and* $2$ *respectively. So, the request of A is capped because he or she exerted little effort and B obtains twice as much as A because he or she worked twice as hard. The allotments to C and B are equal though C worked harder than B but C gets the requested portion.*

This procedure also allows *generalised reciprocity*, as explained in Appendix C. For, the supplier (supplying group, owner of the store) gives goods to others but does not mind who gets them, as long as people (the beneficiaries, perhaps) worked for the group. The beneficiaries, in turn, in principle do not care where the goods or services come from.

In this model, one has to work in order to obtain goods. Also, asking more goods often implies that somebody else works longer, therefore can gain a higher effort index, and thus obtain more goods. So, the preference has a peak located below the maximum available amount. (A group effort bound may further reduce the position of the peak.)

Participants report each other's effort (labour duration and kind of labour). To prevent overreporting, there is an upper bound to the effort: overreporting would imply that fewer time is left in case some unexpected effort needs to be reported; few would be willing to work when the effort is not fully accounted for. (The effort is capped for a period of short duration, but all effort is eventually accounted for in longer periods, so nobody works for nothing, except when surpassing all labour time bounds).

The bound to the reported effort is estimated from the expected effort, but the report of this estimate may be untruthful. The weights are relative: if everyone works twice as hard then the awards stay the same. The main idea is that overestimating effort is no use because the total estimate is distributed over the group; so, when overreporting the effort according to the overestimated effort, everyone would have to work harder but not gain any weight for the goods distribution. So, there is no pointless competition for the sole purpose of raising the effort index (the 'income'). However, the determination of the bound is not easily made (group) strategy-proof as a recurring problem is collusion, which is treated from Section 2.1.6 onward. This will lead to considering an alternative method for faithful representation of reality instead of preferences: inspection of work or the threat thereof.

In the following, this idea is developed further. Several variations emerge, each having disadvantages.

## 2.1 Distribution Basics

Following are some elements common to every variation considered below.

### 2.1.1 Classification of Goods, Time, and Labour

*Goods* are commodities, energy, and the like, but not *services*, which are activities that modify or (help to) produce something, such as baking and cleaning the bakery.

Goods are partitioned into *kinds of goods*, each kind having a time period at the end of which the good will be distributed.

These *time periods* have fixed duration and all periods of a certain length for several kinds of goods start at the same time for each group. For example, daily bread and monthly car distribution.

Work is also partitioned into *kinds of labour* and each kind is equipped, for each time period, with the (physico-mental) *maximum labour duration* a healthy person can work while being able to recuperate, physically or mentally, within a certain (yet unspecified) amount of time. A scientific committee would have to determine these times per area, season, time of day, and so on. For example, daily maximum labour durations are 4 hours open-air mining in the tropics and 6 hours of monotonous moving-band work.

An *effort*, *effort index*, or *labour intensity* is the labour time divided by the maximum labour time. A labour time must always be registered along with the labour type to compute the effort.[1] To continue the example, 2 hours moving-band work (effort 2/6) and 2 hours tropical mining (effort of 2/4, equivalent to 3 hours moving-band work) totals to 5/6. So the efforts can just be added. To prevent working longer than the maximum, labour times are unconditionally truncated, which implies that the sum of efforts is truncated at 1. The mathematics of this implication are not reproduced here.

[1]When registering labour time, the maximum labour duration is not entered directly but via the registration of a labour type, which determines the maximum labour duration. This avoids underreporting the maximum duration.

### 2.1.2 Goods Distribution

As motivated above, goods are distributed using fixed-path rationing. The so-called $N$-path of a participant is a claims guarantee (claims below it are honoured, claims above it return at least this guarantee) but this $N$-path is a (fixed) function of some parameter. For now, this function is multiplication by a weight which is set equal to the accounted effort (a number between 0 and 1). This accounting is treated below but, as mentioned before, unaccounted effort is still valid in longer periods. The weights are scaled such that they add to 1 but this scaling is not necessary: see Appendix A.1. All in all, an effort has a 'value' that depends on the length of a period, the efforts of others, and therefore, even the calendar time.

Notice that no effort means no goods in this basic model. Further, an effort index not used for goods distribution in one period can be used as *savings* for later-on, as when one does not work, but up to 1, the physico-mental maximum effort. Some people (for example, the chronically ill who have not accumulated effort indices) may get priority and for each priority, fixed-path rationing is applied. See Appendix B.

**Question 2** *Are there alternative ways for giving priority? Can the $N$-path process an effort index other than by multiplication with a weight for a fair distribution? Or should fixed-path rationing be generalised to sequential allotment [5]? Whatever the case, who is entitled to determine and enter the priorities?*

In the sequel, suppose it is fair to let the fixed-path distribution incorporate the labour efforts.

### 2.1.3 Labour Distribution

Each participant can have several roles: Employee or *assistant* $h$ ('helper'), who works; employer or *client* $c$, who benefits from this work; and *attestor* $a$, who registers the effort, be it as a client or on behalf of others.

Participants may feel forced to accept jobs in order to collect an effort index but they should be able to change jobs (under the usual jurisdiction). Effort for studying or practicing may also be registered. If multiple employees offer to do a job, the employer will select the employee having the best combination of expected quality of work and labour duration. If there are too few employees to do the same kind of work for multiple candidate employers, then the employees can select the employer offering the best combination of properties, such as willingness or ability to report the longest possible labour duration.

**Question 3** *Is there a fair mechanism, including deliberation, for matching employers and employees that replaces such a 'free labour market'? If there are not enough candidate employees, should the scarce effort index (for expected work) also be distributed by fixed-path rationing for employees who are more or less exchangeable? This may be possible both for continuous and discrete lapses of labour time.*

### 2.1.4 Distribution Groups

A *distribution group* is a limited number of people who work and obtain goods based on their effort from the fixed-path rationing. A more or less fixed group causes preferences for goods to have a peak below the maximum obtainable amount of goods, as expounded in Section 2.1.5. People outside a group may also work for the group or obtain goods from the same fixed-path rationing but only if certain conditions are met, as investigated in Section 4.

To let bounds on labour time suppress collusion, the group should cover a wide range of professions. (This condition has not been formalised yet, nor have arrangements for joining or leaving a group.) Working in a group of rather constant composition will also make a good reputation important or more generally, allow for cooperation. Finally, a group-based economy has the advantage that is can be introduced stepwise.

### 2.1.5 Single-Peaked Goods Preferences

The preferences for goods have a single peak but the peak need not be the maximum a participant could possibly get: the more goods one asks, the more labour is needed for their production, but this labour generally is done by others, who therefore collect more weight for the fixed-path rationing. So, the most preferred amount may be less than the maximum imaginable amount, but this peak will be hard to estimate. In one variation (Section 3.1.3) an upper bound to the group's labour time restricts the duration for registering the duration of unexpected work: asking more goods, hence, more labour time, further restricts this duration, so that also moves the peak away from the maximum possible amount.

**Question 4** *Can participants reasonably estimate the peak?*

Estimation aside, ordering goods but not fetching any because one did not work, bears the risk that others have a greater effort index for obtaining other kinds of goods.

### 2.1.6 Effort Registration and Collusion

Of course, any assistant could easily overreport his or her own effort. Therefore, let the effort be registered based on independent inspection or judgement. To that end, the attestor (usually, the client) registers the effort of the assistant. The attestor can not always inspect but the assistant may also not know what the attestors knows.

**Question 5** *Is there an incentive for the assistant to report the true effort himself or herself, or to report it truthfully to the attestor? The latter case is the problem of the principal.*

The assistant and the attestor may use the registration to *collude*, that is, the attestor overreports the labour duration (or the kind of labour) of the assistant and the assistant reciprocates this favour. For example, the assistant gives the surplus of the goods obtained by the overreporting to the attestor; or the assistant overreports labour of the attestor done for the assistant in the role of client. These types of collusion are defined more precisely in [10].

When estimating bounds to the effort, the candidate assistant and the candidate client may prepare such collusion (between assistant and client) by colluding in a similar way.

Without collusion, and even without bounds to the effort, a client will not profit directly from reporting the effort of an assistant at all (let alone more than the real labour time) because it allows the assistant to obtain more goods. However, underreporting the effort will give the client a bad reputation. This problem is similar to 'not paying the bill' and may be addressed likewise.

To summarise, collusion when reporting effort is suppressed by bounds, of which the determination, however, will turn out to be susceptible to collusion.

## 2.2 Distribution Extensions

The above basics are used to explore variations for a single group below. Prior to that, the following additional requirements for distribution should reduce the number of variations, also for interaction between groups.

### 2.2.1 Distribution of Half-Products

Products are composed of secondary products which in turn consist of, or require, tertiary products, and so on. Consider the case that a single producer produces the primary product (bread) at once in great quantities to be distributed amongst several customers. Grain is needed and grinding the grain requires motor oil, the tertiary product. Before the distribution of bread, there would have to be a distribution of grain based on each customer's effort (weight) for bread, but this requires the distribution of oil, based on the same weights. One problem with this procedure is the enormous delay. Any rationing must compare demands and therefore, wait until a pool of demanders is formed.

**Question 6** *Can production anticipate demands? Would that require that the attestor inspects the store to ascertain that goods must be produced ahead because of expected demand, not just to gain an effort index? Or would some reallocation rule [41] be of help?*

### 2.2.2 Goods Distribution to Intermediaries

People who want bread should not need to apply separately for grain. It would be more practical if the baker merged their requests of amounts of grains to a single amount of grain.

In principle, the baker would apply for grain (produced in the same group) on the basis of his or her own effort, which primarily will be for baking bread. On the one hand, the baker is motivated to exert effort simply to stay in business. On the other hand, this would make the customers dependent on this effort, so they will insist someway that the attestor overreports the baker's effort, thus encouraging fraud. (If the baker needs the grain from another group, he or she probably has no effort index to use for the distribution in that group. This is addressed below.)

**Question 7** *Which joint claim and joint weight combinations are most likely to let an allotment be close to the sum of allotments from separate claims and weights? Or should the fixed-path rationing be replaced with a mechanism that allows combinations which award an allotment close to the sum of individual allotments? Appendix A.3 poses the question more specifically and gives some illustrations. For the case where participants have initial endowments, see [10]. In view of Question 20 on page 24, this question may also become not applicable.*

### 2.2.3 Goods Reuse, Maintenance, and Reclaiming

Maintenance and recycling of goods in one's possession is encouraged as follows.

Maintenance of an object is work one does not only for oneself but also for future users of the object. The object has no inherent economic value, in particular, it does not 'store' any value of the effort, a value which is uncertain anyway. For example, someone has obtained a house for only 0.001 effort index, but it would cost the group a year to rebuild if not maintained properly (not accounting for the time to manufacture the half-products.) If the house becomes available, the future owner may have exerted effort 0.999 for it. By letting the maintenance effort be attested by someone else, maintenance is encouraged to the benefit of future users. The problem still is, that nobody may schedule maintenance because the efforts are accounted relative to each other, that is, the weights in the distribution stay the same.

As to recycling: Goods to be distributed are in a store, possibly in multiple locations. Each store has an attestor who reports the time for bringing or preparing the good, perhaps even for having maintained it. This also prevents that people produce the good, have their labour effort reported, but do not bring the goods to the store. A 'civil servant' might collect goods for the benefit of the community or future generations.

**Question 8** *Do stronger incentives to maintain and recycle exist?*

Reclaiming goods is also an option: A consequence of basing fixed-path rationing on weights equal to the effort index is as follows. If there is only one person applying for the good, this person does not need to have exerted any effort. This is fair unless it occurs that shortly after, other people need the same good. For example, someone urgenty needs a good and therefore, no long-term allocation is run (the object was identified as having a one-second period allocation cycle.) An hour later, someone else also urgently needs the good. The good is now supposed to have been lent and is allocated after all on the basis of efforts. It may be reclaimed but it depends on circumstances whether it can and may be returned. If the interval is not an hour but months or years, one of the potential problems is large-scale registration of goods and their owners.

**Question 9** *Would the expression 'benefit of the community' and lending become pretexts to inappropriately reclaim goods from their owners?*

### 2.2.4 Effort Index Utilisation

Someone who exerted great effort will profit from the same effort in the distributions for every kind of good, where the labour period is measured for the duration that coincides with the period of the goods distribution. Such multiple profit from the same effort may be considered unfair, but it should also be borne in mind that application to one type of good may still be a request for various subtypes. A solution would seem to be deleting the labour time once it has been used as a weight in the distribution; then participants must make a (difficult) decision which amounts of effort to 'pay' for the goods; moreover, the weight would no longer be exogeneous because it would be determined along with the goods requirements.

**Question 10** *To what extent can fixed-path rationing be extended to the simultaneous distribution of various kinds of (infinitely divisible) goods? Such an extension exists for the uniform rule [1, 27] and for sequential allotment [9] (which for at least one kind of good can model fixed-path rationing).*

Participants can shift the effort index to later periods, not only for vacation but also to have the largest weight in the distribution of all goods in a certain period.

**Question 11** *Is such extreme reuse of the effort index by time-shifting improper and if so, how can it be avoided?*

# 3 Design Variations

The above design is now elaborated in various ways.

## 3.1 Effort Bounds

To suppress the aforementioned collusion when estimating and registering effort, consider imposing an upper bound to the effort (that is, the effort index) for each time period and for periods of different length. (Whose effort exactly is bounded, is investigated below.) The weights in the distribution for a certain period will only contain (account for) the effort below that bound of the corresponding period: this weight is the *accounted effort.* The remaining effort is accounted for in a longer period for the part that it overlaps with the labour period, at least, if the bound of that period allows so. There should be such a next bound to avoid overreporting despite capping the effort for shorter periods.

The accounted effort is awarded at the end of the labour period according to some formula, of which the exact nature depends on the situation. Without such formula, the accounted effort would be computed in the course of time, that is, in the order of registration of the effort. So, the earlier the work would be completed, the greater the accounted effort would be. This would force people to have their work completed and registered as soon as possible, phenomenon which could be called 'labour rush.' This obviously is undesirable in many respects.

As briefly explained above, the upper bound discourages overreporting the effort because overreporting leaves less time for finding assistance in case of emergencies, that is, unexpected, important work. (Also, it prevents labour for nothing else than earning an effort index but without any other purpose.) For example, a plumber is suddenly urgently needed but is not willing to work because only a portion of the effort is accounted for, even though the plumber can use the truncated effort index for a longer period. This is why a variety of professions should be represented in a group: such a variety increases the probability that 'emergency workers' are among them.

The bounds can be fixed or estimated, as expounded below. First, some notation is needed.

### 3.1.1 Notation

For any vector (or column) $v$ introduce $\mathbb{1}v = \sum_i v_i$, the *total* of $v$. It can be viewed as the inner product of the row $\mathbb{1} := (1, 1, \ldots, 1)$ and $v$. For matrix $m_{rc}$ (where $r$ is the row index and $c$ the column index) define $\mathbb{1}_c m := \sum_r m_{rc}$, the *column total* of $m$. So $\mathbb{1}_. m$ is a row. Define $\mathbb{1}^r m := \sum_c m_{rc}$, the *row total* of $m$. So $\mathbb{1}^\cdot m$ is a column. Let $\mathbb{1}\mathbb{1}$ be an operator defined by $\mathbb{1}\mathbb{1}m = \sum_{rc} m_{rc}$, the *grand total.*[2]

### 3.1.2 Fixed, Exogenous Effort Bounds

Let $e_{hc}$ be the *effort* exerted by assistant $h$ for client $c$. Let $\hat{e}_{hc}$ be the *reported effort.* To start, consider an individual bound $U = u_h$, independent of $h$, on the *total reported individual effort* $\mathbb{1}^h \hat{e} = \sum_d \hat{e}_{hd}$ of any assistant $h$, where the sum is over the clients $d$ whom $h$ worked for. (Truncation above the maximum physico-mental effort is omitted henceforth.) If the magnitude of $U$ is not of one's liking, one could join another group or create one with a more suitable individual upper bound. However, it will be hard to tell what is a realistic magnitude, the magnitude would have to be determined by a single person (who creates the group) and different times of year or month require different upper bounds to the effort.

For a group of possibly variable size $n$, these problems are also present if a bound $Un$ is imposed on the *reported group effort* $\mathbb{1}\mathbb{1}\hat{e} = \sum_{id} \hat{e}_{id}$.

### 3.1.3 Estimated Effort Bound

As a fixed effort bound is problematic, let a bound on the effort be estimated. Such bounds are best estimated by assistants (rather than clients) because they are familiar with the kind of work. Therefore, each assistant $h$ makes an honest *effort estimate* $f_{hd}$ of the effort $e_{hd}$ for various

[2] The expression '$\mathbb{1}\mathbb{1}$' should be conceived as a single symbol because the ambiguity in the notation $\mathbb{1}\mathbb{1}$ can not be resolved, as is clear from the following. Let $\mathbb{1}'$ be $\mathbb{1}$ as a column instead of a row. So $m\mathbb{1}'_r = \sum_c m_{rc}$. Let $\mathbb{1}^\cdot$ be redefined by $\mathbb{1}^r m := m\mathbb{1}'_r$. So $\mathbb{1}(\mathbb{1}^\cdot m) = \mathbb{1}(m\mathbb{1}') = \sum_{rc} m_{rc}$. However, neither $\mathbb{1}(\mathbb{1}_. m)$ nor $(\mathbb{1}^\cdot m)\mathbb{1}'$ are defined so the brackets can not be omitted.

candidate clients $d$, multiple jobs for $d$ by $h$ taken together. The *reported effort estimate is* $\hat{f}_{hd}$.

Let $\mathbb{1}^h f := \sum_d f_{hd}$ be the *total reported effort estimate*, where the summation is over clients $d$ whom $h$ worked for. If $h$ is honest, then $\mathbb{1}^h f$ obeys a preference relation (weak order) which has a single peak $p_h$ at the honest estimate. The weak order expresses the preference for the objectively best ('honest') estimate. When dishonest, $\mathbb{1}^h f$ follows a preference relation with a single peak 1 at the highest possible effort, which is less than or equal to the physico-mental maximum effort. $h$ may strategically misreport the estimate as $\mathbb{1}^h \hat{f}$, which may be less than the peak (the highest possible effort 1) so as not to raise suspicion or for other reasons, depending on the mechanism. Let $x_h$ be the *award*, which is based on the accounted effort of $h$.

There is a unique type of efficient, symmetric, and strategy-proof mechanisms, the generalised median rule, which selects the median social alternative from a set of alternatives after addition of a number of phantom alternatives, provided preferences are single-peaked. Let the so-called social alternative be the row $p = (p_h \mid h)$, where $p_h$ are honest estimates and the $h$ runs over the group's assistants, who are ordered such, that $p$ obeys a so-called leximin ordening. The median rule does not seem to apply for two reasons. First, if $\hat{p}$ is a misreported alternative, it would be strategically selected such that the outcome is nearest to the honest ('true') estimate. However, a dishonoust agent has no such preference. Second, the median rule applies when an endowment $E$ is to be distributed; by choosing a single phantom row $(E/n, \ldots, E/n)$, the uniform gains rule (the uniform rule for demand exceeding $E$) results [32]. However, in the case of estimating efforts, the total estimate $E$ is not exogenous.

**Question 12** *Can a generalised median rule be made to apply and allow (group) strategy-proof and/or collusion-free estimation of efforts? Or does an alternative mechanism exist that elicits honest estimates, for example one based on needs instead of single peaked-preferences or on (the threat of) inspections of the real effort? (See Appendix E for trial need-based mechanisms.) Does the revelation principle [25, p. 291] apply?*

Answering this question requires a definition of the accounted effort $x_h$. Such a definition is given in the following but that does not answer the question. Once $x_h$ is defined, it will become apparent that collusion other than just group-strategy-proofness necessitates adapting the present approach.

Suppose the total reported effort estimate $\mathbb{1}^h \hat{f}$ were a bound to the labour of $h$. So $x_i = \min\{\mathbb{1}^h \hat{e}, \mathbb{1}^h \hat{f}\}$. If the total reported effort $\mathbb{1}^h \hat{e}$ of $h$ approaches the bound $\mathbb{1}^h \hat{f}$, then $h$ would mind if some client $d$ overreported the effort $e_{hd}$ to the effect that the bound is surpassed. (Ignore the fact that unaccounted effort is accounted for during longer periods.) So, the bound discourages overreporting the effort (as a first step in collusion for such reporting). However, $h$ may simply overreport the bound $\mathbb{1}^h \hat{f}$.

Therefore, consider a bound on the reported group effort $\mathbb{1}\mathbb{1}\hat{e} = \sum_{id} \hat{e}_{id}$. Let the *group effort bound* be $\mathbb{1}\mathbb{1}\hat{f} = \sum_{id} \hat{f}_{id}$. The $\mathbb{1}^h \hat{f}$ will turn out to be a so-called soft bound.

The first purpose of the bound $\mathbb{1}\mathbb{1}\hat{f}$ is to suppress any overreported effort $\mathbb{1}^h\hat{e}$: Suppose that the group's reported total effort so far is $Q$; that $h$ worked $e_{hc}$ for $c$ in reality; and that $Q + e_{hc} < \mathbb{1}\mathbb{1}\hat{f}$ but $Q + \hat{e}_{hc} > \mathbb{1}\mathbb{1}\hat{f}$. That is, if client $c$ overreports, then the group bound is exceeded. Consequently, if $c$ needs an assistant for emergency labour, then no assistant will have his or her effort accounted for in full for the present period; moreover, if the assistant needs emergency labour, he or she is in the same position as the client. This, again, is why several professions should be present in the group.

The second purpose of the group effort bound $\mathbb{1}\mathbb{1}\hat{f}$ is to suppress any overestimated individual effort $\mathbb{1}^h\hat{f}$ by letting an overestimated group effort $\mathbb{1}\mathbb{1}\hat{f}$ be distributed more or less over all participants, who therefore all can work harder (which includes working longer). If most would decide not to work harder but the rest would, then this rest would profit from greater weight in the goods distribution because the accounted efforts are reckoned relatively. So, everyone will try to work harder only not to get behind but possibly without any other purpose. The overall effect, however, is that the extra work may not substantially change anybody's weight for the goods distribution. All in all, the most effective strategy for all would be not to overestimate effort. The following is an investigation into how an individual assistant can be discouraged to yet overestimate.

The way individual efforts are capped so as to distribute the group effort bound is explored as follows. Suppose the total reported group effort exceeds the group effort bound: $\mathbb{1}\mathbb{1}\hat{e} > \mathbb{1}\mathbb{1}\hat{f}$. If only the last reported efforts were capped, then the 'labour rush' would result, so individual efforts must be capped using a formula at the end of the labour period.

To determine that formula, consider the following. As $\mathbb{1}^h\hat{f}$ should not be an individual bound (as established in the beginning of this subsection), $x_h > \mathbb{1}^h\hat{f}$ should be possible. Let the reported estimated group effort $\mathbb{1}\mathbb{1}\hat{f}$ be a (hard) bound to the group's total accounted effort: $\mathbb{1}x \leq \mathbb{1}\mathbb{1}\hat{f}$. Further, the accounted effort $x_h$ for no $h$ can exceed the reported (possibly true) effort $\mathbb{1}^h\hat{e}$, that is, $x_h \leq \mathbb{1}^h\hat{e}$, because it would be unfair if the accounted effort were more than the reported, and possibly true effort. In case $\mathbb{1}^h\hat{e} \leq \mathbb{1}^h\hat{f}$ for all $h$ in the group, then $x_h = \mathbb{1}^h\hat{e}$ because all stayed below their estimate. The surplus estimate $\mathbb{1}\mathbb{1}\hat{f} - \mathbb{1}\mathbb{1}\hat{e}$ is not distributed over all because $x$ already is at its maximum value. In case $\mathbb{1}\mathbb{1}\hat{e} > \mathbb{1}\mathbb{1}\hat{f}$ then someone gets strictly less than the reported (possibly true) effort: $x_h < \mathbb{1}^h\hat{e}$ for some $h$. (For, otherwise $\mathbb{1}x \geq \mathbb{1}\mathbb{1}\hat{e} > \mathbb{1}\mathbb{1}\hat{f}$, contradicting $\mathbb{1}x \leq \mathbb{1}\mathbb{1}\hat{f}$.) That poses a risk, especially if $x_h$ is far less than $\mathbb{1}^h\hat{e}$, and for that reason, $\mathbb{1}^h\hat{f}$ is called a *risk boundary*. To reduce this risk, for all $h$ guarantee that $x_h = \mathbb{1}^h\hat{e}$ as long as $\mathbb{1}^h\hat{e} \leq \mathbb{1}^h\hat{f}$. So, one's effort is fully accounted for whenever it is less than one's estimate. The reported and possibly true effort $\mathbb{1}^h\hat{e}$ can be considered as a *claim*. Therefore, the general name for $\mathbb{1}^h\hat{f}$ is *claims guarantee*. All in all, $\mathbb{1}^h\hat{f}$ is both a risk boundary and a claims guarantee, and therefore goes by the more neutral name of *soft bound*. The accounted effort of $h$ therefore basically is:

$$x_h := \begin{cases} \mathbb{1}^h\hat{f} + \cdots & \text{if } \mathbb{1}^h\hat{e} > \mathbb{1}^h\hat{f} \text{ and } \mathbb{1}\mathbb{1}\hat{e} > \mathbb{1}\mathbb{1}\hat{f} \\ \mathbb{1}^h\hat{e} & \text{otherwise} \end{cases} \tag{1}$$

Suppose the dots were zero. Consider assistant $h$ whose claim exceeds the soft bound: $\mathbb{1}^h\hat{e} > \mathbb{1}^h\hat{f}$. As long as $\mathbb{1}\mathbb{1}\hat{e} \leq \mathbb{1}\mathbb{1}\hat{f}$, the accounted effort is the full effort: $x_h = \mathbb{1}^h\hat{e}$. As time goes by, $\mathbb{1}\mathbb{1}\hat{e}$ increases until it exceeds the group bound: $\mathbb{1}\mathbb{1}\hat{e} > \mathbb{1}\mathbb{1}\hat{f}$. At that moment, $x_h$ suddenly would drop to $\mathbb{1}^h\hat{f}$.

**Example 2** *Let* $\mathbb{1}^{\cdot}\hat{e} = (3,4)$ *and* $\mathbb{1}^{\cdot}\hat{f} = (2,5)$ *so* $\mathbb{1}\mathbb{1}\hat{e} = 7 = \mathbb{1}\mathbb{1}\hat{f}$. *Although* $\mathbb{1}^1\hat{e} = 3 > 2 = \mathbb{1}^1\hat{f}$ *still* $x_1 = 3$. *However, agent* 2 *now works a little harder,* $\mathbb{1}^2\hat{e} = 4.001$. *So* $\mathbb{1}\mathbb{1}\hat{e} = 7.001 > 7 = \mathbb{1}\mathbb{1}\hat{f}$ *and* $x_1$ *suddenly becomes* 2.

For a smooth transition, the dots should be other than zero. Let the *suppliers*

$$S := \{i \mid \mathbb{1}^i\hat{f} - \mathbb{1}^i\hat{e} > 0\}$$

be the indices of the *surplus* $\sigma_i := \mathbb{1}^i\hat{f} - \mathbb{1}^i\hat{e}$ from the overestimation for $i$ in $S$. Let the *demanders*

$$D := \{i \mid \mathbb{1}^i\hat{f} - \mathbb{1}^i\hat{e} < 0\}$$

collect the indices of the *demand* $\delta_i := \mathbb{1}^i\hat{f} - \mathbb{1}^i\hat{e}$ from the underestimation for $i$ in $D$. Distribute any total surplus $\mathbb{1}\sigma$ over any demanders. The reason is not that all surplus estimates should be used because such surplus is 'wasted' if everyone's effort is below the estimate and accounted for exactly. One reason for distribution is that the transition between $\mathbb{1}\mathbb{1}\hat{e} \leq \mathbb{1}\mathbb{1}\hat{e}$ and $\mathbb{1}\mathbb{1}\hat{e} > \mathbb{1}\mathbb{1}\hat{f}$ is smooth, as proven in Appendix G. The other reason is that (without collusion) assistants are encouraged to honestly estimate their efforts. This is seen as follows. First, consider overreporting the estimate $\mathbb{1}^h\hat{f}$, that is, $\mathbb{1}^h\hat{f} > \mathbb{1}^h f$. It more probably is an overestimation of the registered effort ($\mathbb{1}^h\hat{f} > \mathbb{1}^h\hat{e}$) than when honoustly estimating, $\mathbb{1}^h\hat{f} = \mathbb{1}^h f$, so probably, the surplus $\sigma_i$ is distributed over others, who will get more goods in the fixed-path distribution. There is a lower probability that it is an underestimation still, $\mathbb{1}^h\hat{f} < \mathbb{1}^h\hat{e}$, than when for an honoust estimate, $\mathbb{1}^h\hat{f} = \mathbb{1}^h f$, but if that happens, then the demand must be met by supply from others. Second, consider underreporting the estimate, $\mathbb{1}^h\hat{f} < \mathbb{1}^h f$. The above cases are the same but with probabilities reversed. So, $h$ is more likely to be in demand than to supply some effort index. Both cases are not advantageous. So, this mechanism would be stratege-proof to misestimation of effort. Further considerations are in the 'planning game' described in Appendix F.

The effort bound needs to include an estimate $U$ for unexpected effort, that is, of work to be done for client $c$ by a yet unknown assistant $h$. If $U$ were too large, it would not deter attestors from overreporting the effort $\mathbb{1}^h e$ of an assistant $h$ because the attestor may turn out to be a client needing an assistant for unexpected work; instead, it would invite everyone to exert more effort, or at least let more effort be registered. Therefore, each potential client $c$ estimates the 'expected unexpected' effort $N_c$ to be needed and $U := \mathbb{1}N$. Alternatively, each candidate assistant $h$ estimates such effort $X_h$ to be exerted and $U := \mathbb{1}X$. In both cases, $U$ may be unrealistically high.

**Question 13** *How can the unexpected effort of a group for a certain time interval be realistically (so, honestly) be estimated, in particular, if no statistics are (yet) available? See also Section 3.3 on funds.*

Let

$$\alpha := \frac{U + \mathbb{1}\mathbb{1}(\hat{f} - \hat{e})}{\mathbb{1}\mathbb{1}(\hat{f} - \hat{e})|_D}$$

then a formula would be:

$$x_i = \begin{cases} \mathbb{1}^i\hat{f} + (\alpha - 1)(\mathbb{1}^i\hat{f} - \mathbb{1}^i\hat{e}) & \text{if } \mathbb{1}^i\hat{e} > \mathbb{1}^i\hat{f} \text{ and } \mathbb{1}\mathbb{1}\hat{e} > U + \mathbb{1}\mathbb{1}\hat{f} \\ \mathbb{1}^i\hat{e} & \text{otherwise} \end{cases} \tag{2}$$

This is proven in Appendix G where the claims are $c_i = \mathbb{1}^i\hat{e}$, the soft bounds are $\mu_i := \mathbb{1}^i\hat{f}$, and the endowment is $E := U + \mathbb{1}\mathbb{1}\hat{f}$.

Although this formula is applied at the end of a time period, assistants may well predict that the group bound is soon to be exceeded because efforts indices are public. Although a labour rush is avoided, these dynamics allow the following estimation game.

Consider collusion when estimating the efforts. Suppose everyone estimates honestly except participants $i$ and $j$, who use dishonest overestimations $\hat{f}_{ij}$ and $\hat{f}_{ji}$; participant $j$ promised to $i$ to enter an overreported effort $\hat{e}_{ij}$ and vice versa. (Other reciprocations are also possible, for example, share in the extra goods thus obtained.) So, $\mathbb{1}^i\hat{f}$ and $\mathbb{1}^j\hat{f}$ are also overestimations. As long as the group's bound is not exceeded, $\mathbb{1}\mathbb{1}\hat{e} \leq \mathbb{1}\mathbb{1}\hat{f}$, assistants $h$ may report more than estimated, $\mathbb{1}^h\hat{e} > \mathbb{1}^h\hat{f}$, and still have an accounted effort index $x_h = \mathbb{1}^h\hat{e}$ according to Equation 2 for $U = 0$. These $x_h$ compete with $x_i$ and $x_j$ so $i$ and $j$ willl report $\hat{e}_{ji}$ and $\hat{e}_{ij}$ high enough to let $\mathbb{1}\mathbb{1}\hat{e} > \mathbb{1}\mathbb{1}\hat{f}$. Then all accounted effort $x_h$ drop, be it smoothly, also for $h = i$ and $h = j$, but there loss is partially imaginary or even completely if they have not worked at all. However, would attestors be truthful (or one of the colluders defect by not registering the promised effort) then an overestimated effort is not used in full and the excess may be used for accounting more effort of those who worked more than estimated.

Can three or more players be deterred from collusion by increasing the 'punishment' for one of the colluders if the other defects, that is, not live up to the deal to collude? If the surplus $\hat{f}_i - \hat{e}_i$ is given to reporter $j$ then it would seem to be attractive for $j$ to defect, that is, report far less than initially convened, as in the prisoner's dilemma. The distribution functions $\psi$ and $\rho$ in Appendix A.1 and Appendix G.6, respectively, might take care of such arrangements. This arrangement invites underreporting and thus discourage collusion, but $j$ may find establishing a good relation with $i$ more important (in order to repeat the collusion) than defecting. See the planning game in Appendix F for some additional considerations. A disadvantage is that honest overestimation would equally be punished.

In conclusion, this 'overestimation game' provides no guarantee that effort estimates are honest. The underlying reason may be that labour estimates can be made 'out of thin air.'

**Question 14** *Does a procedure exist to elicit the true effort estimates and avoid collusion? A Groves mechanism may choose the group effort bound as a 'social alternative' but such mechanisms generally are susceptible to collusion. Should this cooperative game be cast to a non-cooperative game first? (The starting point may be [34] according to [36].)*

Another disadvantage of a group effort bound is that group members, even in small groups, may imagine that, if the group effort is close to its bound, they will still be able to interest an assistant for emergency work, a kind of 'tragedy of the commons'. In the next variation, it is entirely one's own responsibility to keep away from the bound.

## 3.2 Effort Grants

As estimated bounds do not avoid 'overestimation games' such as in Section 3.1.3, consider distinguishing between the reported effort index (labour intensity) and an effort index to report it with. Let the *effort grant* (or *effort allowance*) be an effort index granted to a client $c$ with which he or she can report the effort (labour duration and labour type) of the assistant; once the effort is reported, that effort goes off the effort grant, as if it were a purse of money to pay with. There is an effort grant for each period and for periods of various lengths. The reported effort cannot be used as an effort grant, so it is not a transferable entity, as is money. Collusion in this model can be made somewhat unattractive by the threat of inspections, as set out below.

Generally, a bound on the labour may be an obstacle to ambitious projects. For, a fixed bound may leave no room for the project; if assistants estimate the bound, the must agree with the project in advance by making an estimate at all.

**Question 15** *Can the effort grant be turned into an effort index which is (perhaps partially) earned? For example, by collecting earned effort in an 'effort purse' (or 'earnings') with which to register the effort of others. (The earned effort in the purse can no longer be used as weight in the fixed-path distribution.) This may also circumvent the problem of collusion when estimating the effort grant (or group effort bound). The dynamics are yet unclear because the earned effort would be bounded by previously earned effort. Moreover, the bound becomes more 'sloppy' (some may amass earnings) and might allow overreporting and even the equivalent to 'wage slavery'. The idea of an 'effort purse' recurs when discussing group interaction, Paragraph 4.3.4.*

For now, the focus remains on bounds.

### 3.2.1 Direct Effort Grants

As before, $\hat{f}_{hc}$ is the effort for $c$ estimated and reported by $h$. The estimate may not be honest. The simplest determination of the effort grant of client $c$ is $\mathbb{1}_c\hat{f} = \sum_h \hat{f}_{hc}$, the total reported estimate of the effort to be spent for $c$, where $h$ runs over the assistants who will work for $c$. So, the effort grant is like 'money for free' but time actually is for free too.

First, assume there is no collusion. For group effort bounds (Section 3.1.3), the whole $\mathbb{1}\mathbb{1}\hat{f} = \sum_{id} \hat{f}_{id}$ was distributed over the assistants to discourage overestimation of $f_{hc}$ as a large $\hat{f}_{hc}$. (For, overestimation by one, for example to account for unexpected work, would allow all to work harder but the weights would stay more or less the same.) Here, distribute $\mathbb{1}_c\hat{f} = \sum_i \hat{f}_{ic}$ only over the assistants $i$ who worked for $c$. Let $\mathbb{1}_c\hat{e} = \sum_i \hat{e}_{ic}$ be the effort reported for client $c$. Instead of Equation 1,

Appendix G yields:

$$x_{hc} := \begin{cases} \hat{f}_{hc} + \cdots & \text{if } \hat{e}_{hc} > \hat{f}_{hc} \text{ and } \mathbb{1}_c\hat{e} > \mathbb{1}_c\hat{f} \\ \hat{e}_{hc} & \text{otherwise} \end{cases}$$

The dots are such that $\mathbb{1}_c x = \mathbb{1}_c\hat{f} + U$ where $U$ is an estimate of the unexpected effort, as before: $U = N_c$ (the estimate of effort needed by $c$) or $U = \mathbb{1}X|_{\Gamma(c)}$ for $X_h$ the unexpected effort to exert, as estimated by $h$, and $\Gamma(c)$ the set of assistants of $c$. The unexpected effort is hard to estimate. This raises the following question.

**Question 16** *How to trustworthily register unexpected labour effort within a group? From an 'effort fund'? Should the group bounds on effort be reintroduced for unexpected labour only?*

Now consider collusion. The same 'overestimation game' can be played as in Section 3.1.3 but at a smaller scale, as if in a group with a single client. The simplest case occurs when $c$ and $h$ only 'work' for each other and overestimate $\hat{e}_{hc}$ and $\hat{e}_{ch}$. Three or more assistants of $c$ may also easily collude. In conclusion, collusion is not avoided and again allows to create an effort index 'out of thin air.' Therefore, consider some amendations of this procedure: apportioned and challenged effort grants.

### 3.2.2 Apportioned Effort Grants

Suppose $\mathbb{1}\mathbb{1}\hat{f} = \sum_{hc}\hat{f}_{hc}$ is to be split up in effort grants. Two members may collude by promising each other overreporting and thus allowing effort overestimation $\hat{f}_{hc}$. If one distributes $\mathbb{1}\mathbb{1}\hat{f}$ using the uniform rule (that is, the constrained equal awards rule [39, p. 42]), then everyone would want, and request, the maximum in order to suppress collusion of others when estimating effort or to collude after all by obtaining the largest effort grant. The result will be that all effort grants are equal. That may be unfair to those who really need a large effort grant. In other words, the preferences may be truthfully reported, but the preferences themselves do not reflect honest estimates of the effort.

### 3.2.3 Challenged Efforts

As a potential solution, introduce the *challenge*, that is, the possibility of inspections. This is a proven model if inspections are rare or done by dedicated personnel (Appendix D) but might not work well in this case. The candidate assistants $h$ have reported estimated labour efforts $\hat{f}_{hc}$ for various clients $c$.

Suppose the $\hat{f}_{hc}$ are public, so $\mathbb{1}_c\hat{f}$ is public too. Assume that group member $b$ suspects that $\hat{f}_{hc}$ is higher than really needed, for example, because some assistant $i$ exists who could do the same job faster and just as well. (If $h$ colludes with $c$, then $c$ will not choose $i$ instead, nor can $c$ be forced to accept $i$, because collusion is not sure.) Then $b$ can assign an attestor $a$ or be one himself or herself to record the labour time of $h$. (There can not be two attestors of the same job.) Suppose that the attestor $a$ is different from $b$ and that the choice of $b$ is mediated by a digital system so that the attestor does not know who is $b$ so there is no display of distrust. The attested labour effort, or the lack thereof, is public: if the attestor does inspect, he or she can not be accused of being suspicious because refusal would be public; and if inspection is impossible, this would have to be explained. Without inspection by the attestor, the effort registration must still be based on good faith. If $b$ suspects that attestor $a$ will collude with $h$ then $b$ should try to attest himself or herself, but then $c$ and $h$ will know that $b$ distrusts them. Let $\hat{B}_c := \mathbb{1}_c\hat{f}$ be the effort grant. Define new grants $\hat{B}'_c := \hat{B}_c - \hat{f}_{hc}$ and $\hat{B}'_a := \hat{B}_a + \hat{f}_{hc}$. That is, attestor $a$ gets a portion of the grant of $c$ to 'pay' for client $c$. If the estimate $\hat{f}_{hc}$ is overreported, then attestor $a$ can report more labour effort than when $h$ reported truthfully, in which case the effort of $h$ can still be reported. Contrary to the inspection mechanisms in Appendix D, overreporting poses no quantitive risk: the assistant keeps the prospect of having his or her effort registered. The only disadvantage for $c$ and $h$ is a potential loss of reputation. For $c$ other than claimant or claimee, $\hat{B}'_c := \hat{B}_c$. The benefit of challenges is that the 'overestimation game' is harder to play because in case of collusion, an attestor may replace one of the players and defect, that is, not register the effort that was promised by the other. A disadvantage may be that there are far more colluders than inspectors or that inspections are needed too often.

**Question 17** *Can the revelation principle [25, p. 291] be applied to reveal the true estimated effort instead of having to rely on inspections of the exerted effort, such as the challenge above and in Appendix D? Or a collusion-free combination of the Groves mechanism and inspections [11]?*

### 3.2.4 Stages for Challengeable Effort Grants

The effort grants model is recapitulated by traversal of the following stages.

**Request goods** Group members report the (most preferred) amount of goods of a certain kind and the time interval during which to obtain that amount (by the latest and the earliest). The kind of good determines the phase and frequency of the distribution moments. The good is distributed at the earliest moment in the desired time interval.

**Request labour** Each member creates, in the role of candidate employer $c$, vacancies for production labour and services.

For example, to bake 50 breads, the baker $c$ estimates to be needing 5 people to do various jobs, each 20 hours a week. These jobs are also for baking cakes and other delicacies, but no distinction is made between fixed time (cleaning) and variable time (selling bread to a single client). An attestor reports the labour times of baker and personnel separately on behalf of all clients.[3]

[3] Why not 'pay' only the baker, who redistributes the labour time over personnel? Formally, hours to be paid, after conversion to hours worked, are again used as hours to be paid, a confusion of meaning, inviting hours to become a currency. Worse, there is no bound on these hours anymore, because they can circulate indefinitely, as with money. Finally, it is also very

**Estimate labour time** An assistant $h$ makes an estimate $\hat{f}_{hc}$ of the effort $e_{hc}$ (labour time and kind of labour) to be made for client $c$ and estimates the start time of the labour. These estimates include quotations of labour, some of which may be refused. The estimate is final at the start of the smallest period which encloses the labour period. The effort grant created for $c$ (to 'pay' the assistants) is $\mathbb{1}_c\hat{f} = \sum_h \hat{f}_{hc}$.

**Challenge Effort Estimates** Any person $b$ who suspects that assistant $h$ and client $c$ collude when estimating effort $e_{hc}$ may indicate an attestor $a$ (or attest himself or herself) who registers the effort of $h$. The effort grant $B_c$ is diminished by $\hat{f}_{hc}$ and $B_a$ is increased by the same amount, so the attestor $a$ 'pays' instead of $c$.

**Work and Report Efforts** Each assistant exerts efforts, possibly for unexpected labour. The attestors report these efforts using the effort grants defined in a previous period. (The first period ever does not start with goods distribution.) Long-running labour may have to be partitioned into multiple jobs of which the labour time is to be reported separately. Effort indices may be saved for later use.

**Distribute Goods** The goods are distributed (rationed along a fixed path) at the end of the period, based on the requests and weights which equal the relative accounted effort during the distribution period that holds for the goods, where the accounting is according to the formula for maximal allocation subject to soft bounds, as explained above. The whole effort index is reused for other kinds of goods during the same period and for goods having a longer distribution period.

After the distribution program ran, customers obtain their goods.

## 3.3 Group Funds

As an anticipation of multi-group considerations, consider actual funds. The total estimated effort $U + \mathbb{1}\mathbb{1}\hat{f}$ can be conceived as a fund for the accounted effort, in particular, $U$ for unexpected, emergency work. Two more such funds are considered in the following.

### 3.3.1 Effort Index Funds

The accounted effort indices may be stored in an *effort index fund* to apply for goods that serve the whole group or an arbitrary member of the group, for example, to provide food for ill people who have accumulated an insufficient effort index. The distribution may as well be based on priorities.

**Question 18** *How much accounted effort indices should a person contribute? When should an effort index be donated form this fund and how large?*

impractical to estimate the labour time of personnel: they, too, would want to be paid in total for their personnel, like a babysit, and so on. In particular, what if the babysit works for two servants of the baker? (Duplicate count). Or what if the baker is the babysit? (Loop)

### 3.3.2 Effort Grant Fund

Some of the effort grants $\mathbb{1}_c\hat{f}$, too, can be stored in an *effort grant fund* from which to 'pay' assistants who suddenly must serve the whole group or who could serve an arbitrary member of the group, as with a fire brigade. However, this would simply amount to a construct similar to a total group effort $\mathbb{1}\mathbb{1}\hat{f}$.

# 4 Multiple Groups

Perhaps a choice can be made between the following scenarios: group overlap and interaction between groups.

## 4.1 Group Overlap

Not all goods can be obtained from a single group, so people may be part of several groups. This would also diminish any kind of competition between groups. Would a participant be able to distribute the maximum labour effort 1 as portions over the groups, then choosing a very low such portion would allow to attain an effort of 1 in that group without substantial effort. So, the labour effort must still be computed with respect to the maximum physico-mental labour duration.

## 4.2 Interaction between Groups

Consider groups $A$ and $B$. Suppose client $c$ in $A \setminus B$ is helped by assistant $h$ in $B \setminus A$. For example, $c$ is transported by taxi driver $h$. The effort is registered as $\hat{e}_{hc}$. Conversely, $h$ needs goods from group $A$, for example, lubrication oil. A request can also be made on behalf of consumers from $B$, as when $h$ is a baker. The case that $h$ works for $A$ as a whole in some way is not discussed. The effort spent by $h$ for group $A$ may be his or her own effort or from group $B$.

The procedure by which assistant $h$ participates in the distribution of goods (such as oil) from group $A$ is as follows. Participating in this distribution is an advantage for group $A$ if $h$ contributed to members of $A$ (strict recipocity). As contributions expressed in goods are generally incomparable, impose the condition that $h$ has worked for some $c$ from $A$ (or for $A$ as a whole, but that event is not further discussed). Generally, let the effort of some assistant $i$ from group $X$ be reported as $\mathbb{1}^i\hat{e}$, so the *mean effort* of $X$ per assistant is $\epsilon_X := \mathbb{1}\mathbb{1}\hat{e}/H_X$ where $H_X$ the number of assistants in $X$. Consider the case $\epsilon_B > \epsilon_A$, so assistants from $B$ are expected to work harder than those in group $A$. Then members of $B$ who worked for $A$ would be expected (because of the mean) to have more weight for the distribution of goods from $A$ only because members of $A$ thought working harder than $\epsilon_A$ would not be worth the effort. To avoid competition between the groups, any effort $\sum_{c\in A}\hat{e}_{hc}$ of $h$ spent on $A$ would have to be accounted for $A$ as $\min\left\{\epsilon_A, \sum_{c\in A}\hat{e}_{hc}\right\}$. So, $h$ did not engage in the estimation $\mathbb{1}\mathbb{1}\hat{f}$ of the total effort in $A$ but $h$ may join in for at most the mean effort of $A$. Also, $h$ from $B \setminus A$ may work for a client from $A$ because hard labour is better accounted there or because there is work at all, perhaps badly accounted for the present period.

**Question 19** *Is this procedure fair and non-manipulable? Do superior alternatives exist? For example, just letting assistant $h$ join group $A$ while staying in $B$, that is, overlap of $A$ and $B$ in $h$?*

In the following, two possibilities are considered: effort is kept private or effort is shared with the group. Depending on which possibility is chosen, the interaction between groups as a whole is discussed. The possibilities hold for the model of group effort bounds and the model of effort grants.

**Private Effort Index** Suppose that $h$ adds his or her own effort index $\hat{e}_{hc}$ (where $c$ in $A$, as always) to the accumulated effort indices of $B$, defined as $\alpha_A^h := \sum_{d \in A} \hat{e}_{hd}$. For example, $h$ drives a taxi for $c$ and adds the effort index to the collection of $B$ to be spent in $A$. Later-on, $h$ gets oil from $A$ based on the effort from $\alpha_A^h$.

**Private Effort Index in Group Effort Bound Model** Let $E_A$ and $E_B$ be the bounds on the total effort of the groups $A$ and $B$. In this case, $c$ is willing to overreport the labour of $h$, that is, to have $\hat{e}_{hc} > e_{hc}$ as a 'favour' (without expecting this 'favour' to be reciprocated) because only the total effort of $B$ is brought closer to the upper bound $E_B$ of $B$. However, if the group effort is too close to $E_B$, then this is not a favour, neither to $h$ nor to group $B$, because little time is left to $h$ or others from $B$ for emergency assistance (a plumber, say) and $h$ can not prevent $c$ from overreporting.

**Private Index of Effort in Effort Grants Model** In this case, $c$ will be reluctant to overreport the labour duration of $h$ because the additional effort index goes off his or her effort grant.

In order to apply for grain from $A$, the baker $h$ does not need to have an effort index $\alpha_A^h$ to be spent in group $A$, that is, the baker need not also be a taxi driver for clients $c$ from group $A$. Instead, requests for grain (that is, for bread) and efforts $\alpha_A^i$ from customers $i$ of the baker are merged (Appendix A.3). The disadvantage is that customers would need to have worked for $A$ (to obtain grain) as well as for $B$ (to obtain bread).

**Shared Effort Index** Assume that the effort index of $h$ is added to a *shared effort index* of group $B$ for efforts exerted for clients from $A$. This would allow the group as a whole to apply for the distribution of goods from $A$. The only incentive for $h$ to work for $c$ would be to anticipate that $h$ once will be needing a good from group $A$ and expects that the period of the good during which the effort holds is not over. However, $h$ may also not take the risk of seeing only others profit from his or her contribution. As there is only collective debt, this is another case of generalised reciprocity. The shared effort index should only be used for intermediaries between groups, such as the baker from $B$ who needs grain from group $A$.

**Question 20** *Does the shared effort index circumvent the problem of merging claims and effort indices, as mentioned in Question 7 on page 11?*

**Shared Effort Index in Group Effort Bound Model** If $c$ overreports effort of $h$, then $\hat{e}_{hc}$ is added to the shared effort index of $B$ and there is no risk that the bound $U_B$ of the total group effort index is approached.

**Shared Effort Index in Effort Grants Model** As before, $c$ is reluctant to overreport the effort of $h$ and no bound is needed on the shared effort index to suppress overreporting.

**Question 21** *Should h be 'paid' from a shared effort grant of group A, similar to the shared effort index? How should such a shared effort grant be filled?*

In both cases, the effect of overreporting by $c$ is that $B$ can obtain more goods from $A$ but $c$ may ignore (or not be aware of) this disadvantage for $c$.

**Question 22** *Should a bound on the shared effort index be imposed and if so, how should it be determined and what happens if the bound is surpassed?*

The shared effort index can not be exhausted so it differs from a fund, as described in Section 3.3.

The choice between private and a shared effort index may become not applicable after the following considerations.

## 4.3 Effort Index Transferral

A problem above was that the baker from $B$ needs grain from group $A$ (on behalf of many members of $B$) but may not have exerted effort for $A$, so other members of $B$ should have worked for members of $A$. An additional problem is the lack of a coincidence of wants, illustrated as follows. The taxi driver $h$ from $B$ who drove client $c$ from $A$ does not need goods from $A$ so can not 'spend' the effort index in group $A$. The same holds for production instead of services, for example a potter from $B$ made pots for $A$ but can not use the effort index to obtain something else from $A$.

One solution would seem to abolish groups and let arbitrary persons participate in the goods rationing along a fixed path. However, that would entitle anyone (member of some community or not) who worked somewhere to obtain goods, in other words, members would exert effort only to see the goods go; vice versa, they would produce for no community in particular.

A compromise may be some generalised reciprocity (Appendix C) between groups and their members. The reciprocation should not be a long-term right because such rights may never be executed or these rights start to 'lead a life of their own' as debts having unwanted side-effects. The reciprocation of a good which is allocated (from the store in group $A$) to a non-member (someone from $B$) should not be another good (barter) because goods are incomparable and there may not be a coincidence of wants. The only remuneration for this cross-group donation would be effort that has been exerted for group $A$, but *not necessarily by the recipient of the good*, so possibly someone from a group other than $B$ (and $A$). That is, it does not matter who worked for group $A$ as long as the work has been done, so the weight in the fixed-path distribution for a particular person could be the effort index of anyone who worked for group $A$. Labour instead of goods (but also labour that produces goods) can simply be reciprocated by the registration of the effort. Contrary to an effort index that can be used as a weight only in the group for which the effort was exerted (the favoured group or beneficiary), the effort index must be turned into some general-purpose index, independent of the beneficiary. In conclusion, the effort index would be made independent of the beneficiary and the benefactor. The following is an exploration of some candidate arrangement for such generalised reciprocity.

### 4.3.1 Trading Directed Effort Tokens

As it does not matter who worked for the group to obtain a good, as long as the work has been done, the application to goods distribution might be arranged using 'promissory notes' of the effort index, that is, *directed effort tokens.*[4]

**Example 3** *The taxi driver $h$ from $B$ earned* 1 *hour driving $c$ from group $A$ and the maximum driving (and waiting) time is* 12 *hours per day, so effort index $\frac{1}{12}$ for $A$. The taxi driver does not need a good from $A$. Next, a plumber exerts effort $\frac{2}{36}$ for the taxi driver and is not paid by an effort index $\frac{2}{36}$ for $B$ but by a promissary note of $\frac{2}{3}$ of the effort index $\frac{1}{12}$ of $h$ for work done for $A$. The taxi driver is left with the remaining $\frac{1}{36}$.* The $\frac{2}{36}$ does not go off any effort grant, so the taxi driver uses the effort $\frac{2}{3}$ as if it were a weight in an imaginary distribution of labour instead of goods. *The plumber may need something from $A$ and if not, accept the directed token because he or she expects to find someone who does need a good from $A$. Etcetera.*

Notably, an effort token has a value equal to exerted effort and would only be usuable as a weight for fixed-path distribution. However, now it would also be used for registering effort, similar to the previously discussed resources, that is, the total estimated group effort and someone's effort grant. As these resources are estimated (perhaps dishonestly) they compromise the meaning or value of the effort token, which represents true (or at least registered) effort. Moreover, substracting the $\frac{2}{36}$ from the taxi driver's $\frac{3}{36}$ effort is treating an effort index as a commodity, an independent entity, which is traded. Such trade runs counter to the fact that an effort index has a long-term meaning: for daily distribution in the evening, today's effort no longer holds tomorrow, but for yearly distrisribution, it is still valid. Also, it may be used for multiple distributions at the same time (which is of some concern.)

Eventually, the directed token is used by someone who gets various types of goods from $A$ (after which it perhaps should not be possible to

[4]The word 'effort note' might be conceived as the effort registration.

pass it on further.) This points to another disadvantage: some person can accumulate effort indices for a group (though theoretically not to more than 1) from various people who worked for that group. The person would then have a large, and perhaps unfair, weight in the fixed-path distribution.

### 4.3.2 Undirected Effort Tokens

Despite these advantages, explore a variation. Yet another disadvantage is that a directed effort token may outdate, at least with respect to short-term usage: if one cannot use a directed effort token as a weight in the fixed-path distribution of a particular group, one has to wait for someone who can, or who knows someone else who can, and so on. To speed up, consider removing the indication of the group for which the effort was exerted and call the resulting token an *effort token*. It is simply worth an index of effort exerted for whoever at a certain time (so holding for particular periods) and which can be spent anywhere.

However, such effort tokens would render the system with effort bounds or effort grants partially obsolete, because some effort need not be estimated but can be exerted for whomever turns up with an effort token. To amend this, the effort tokens may be used for unexpected labour only, but that would not facilitate interaction between groups.

### 4.3.3 Directed Effort Token Pool

To stay closer to the original meaning and processing of the effort index, consider that the taxi driver $h$ transfers his or her effort index $\hat{e}_{hc}$ (where client $c$ is from group $A$) to an *effort token pool*, that is, of directed tokens. Anyone (including $h$) interested in obtaining goods from $A$ could use that token.

Automated trading would be as follows. The transferral of any $\hat{e}_{hc}$ to the pool create an open *slot* in group $A$ of the same size: it represents effort exerted for group $A$ which is not necessarily used by $h$. Suppose $i$ from $Y$ exerted effort $\hat{e}_{id}$ (for a particular period) for some client $d$ from group $X$ other than $A$.[5] However, $i$ wishes goods from $A$. (So $i$ is in the same position as $h$ who wants no goods from $A$ but from some $X$.) Let $P$ be the whole population. Let

$$\epsilon_i^A = \max_{(\eta,\gamma)\in P\times A} \min\{\hat{e}_{id}, \hat{e}_{\eta\gamma}\}$$

be the effort index to be used by $i$ in $A$. In words: $\epsilon_i^A$ is the largest effort someone exerted for $A$ but that does not exceed the effort $\hat{e}_{id}$. (This query can be accelerated by first querying $\gamma$ and then $\eta$.) Let $\pi_j^Z$ be the pool of effort by any assistant $j$ for some group $Z$. Let $(h, c)$ be the argument $(\eta, \gamma)$ that defines $\epsilon_i^A$. So $\epsilon_i^A = \hat{e}_{hc}$. The effort indices of $h$ and $i$ are exchanged as follows:

$$\begin{array}{llll} \pi_i^X & = \hat{e}_{id} + \cdots & \pi_h^A & = \hat{e}_{hc} + \cdots \\ \tilde{\pi}_i^X & = \pi_i^X - \hat{e}_{hc} & \tilde{\pi}_h^A & = \pi_h^A - \hat{e}_{hc} \\ \tilde{\pi}_i^A & = \pi_i^A + \hat{e}_{hc} & \tilde{\pi}_h^X & = \pi_h^X + \hat{e}_{hc} \end{array}$$

[5]Of course, '$id$' is not the identity but '$i, d$'.

The $\tilde{\pi}$ is the new pool. So, group $A$ reckons with the same amount of work that has been done for it but it does not matter by whom (generalised reciprocity). It would seem that the slot should now be closed, at least to be able to track changes. A disadvantage is that the exchange causes the pools $\pi_j^Z$ to contain fragmented effort indices ('left overs') of which outdate soon for short periods though they remain valid for longer periods.

The benefit for $h$ to transfer the effort $\hat{e}_{hc}$ to the pool is that the baker from $B$ may utilise this effort when merging requests for grain from $A$ (and $h$ from $B$ is a customer of the baker.) A potential drawback is that $h$ may regret transferring the effort to the pool and after all does want a good from group $A$, but someone else may have occupied the slot. Users may therefore be inclined to keep effort which turns out to be of no use. In conclusion, the effort for other groups, such as $\hat{e}_{hc}$, is *automatically* transferred to the pool in order to facilitate interaction between groups. The major challenge would be to run all queries against a table which is continuously being updated.

### 4.3.4 Effort Purse

In the above, earned effort was equated to effort to be registered. To distinguish between the two concepts again, let earned effort $\hat{e}_{hc}$ be added to the effort grant, which now should be called *effort purse* (for lack of a better term) and replace the above pool. So, $h$ can no longer use $\hat{e}_{hc}$ to obtain goods from $A$ or anywhere else, only to register labour. This lead back to Question 15 on page 18.

# 5 Conclusion and Further Research

There is little use in elaborating details if the basis (for example, fixed-point rationing) is the wrong point of departure. So, the questions in this document should be answered first in order to obtain a coherent multi-group model that does not offer major 'loop holes'. If such is possible, then the next step would be to finish an application for a single group that allows to simulate this economy as a game before users put it to practice. Feedback from users would help to decide whether expansion to a multi-group model is useful.

In conclusion, research and development of this economy can only be a joint effort, ideally in a (perhaps multidisciplinary) project setting.

# A Rationing along a Fixed Path

Let $N = \{1, 2, \ldots, n\}$ be the collection of agents. An $N$-path for agent $i$ is a function $g_i(\lambda)$ of some parameter $\lambda$. (All values are positive real-valued numbers unless stated otherwise.) Define $\mathbb{1}v := \sum_{i \in N} v_i$ for any $n$-vector $v$. For a subset $S$ of $N$ let $v|_S := \big\{(i, v_i) \mid i \in S\big\}$ be the restriction of the function $v$ to $S$. Let $E$ be the *endowment* to be distributed and let agent $i$ have *claim* $c_i$. Consider $\mathbb{1}c \geq E$. Let $\lambda$ be the (unique) solution of

$$\sum_{i=1}^{n} \min\left\{g_i(\lambda), c_i\right\} = E$$

and let the *award* be $x_i := \min\left\{g_i(\lambda), c_i\right\}$. If there is a surplus, $\mathbb{1}c \leq E$, the min becomes max to ensure strategy-proofness of the scarce-goods case. Call this procedure *distribution along a fixed path.*

As a start, take $g_i(\lambda) = w_i\lambda$ for suitable *weights* $w_i$. This yields the method of weighted gains [31, pp. 648, 658]. For the case of surplus, the gains are interpreted as negative (not as losses.) The weights are exogeneous variables, as are the $N$-paths.[6] If $\mathbb{1}g(\lambda) = \lambda$ then $g_i(\lambda)$ and the problem are called *scaled.* The not necessarily scaled is called *non-scaled.* Scaling allows to easily show that also discrete (entire) quantities of commodities can be distributed along a fixed path. (The discrete case illustrates that the distribution may not be anonymous or envy-free.) The scaled fixed-path distribution has been called *fixed-path rationing* but the term 'rationing' is not applicable for a surplus, so this name should become obsolete[7] whereas the distribution need not be scaled by definition when it can be scaled separately, as the following shows.

## A.1 Scaling

Generally, there is no bijection between solutions of the scaled and non-scaled problem, as the following shows.

Let $g'_i(\lambda')$ be a non-scaled $N$-path for any $\lambda'$. Define $\psi'(\lambda') := \mathbb{1}g'(\lambda')/\lambda'$. Scale $g'_i$ as $g_i(\lambda) = g'_i(\lambda)/\psi'(\lambda)$ for any $\lambda$. Conversely, given some function $G(\lambda')$ of $\lambda'$ in the role of $\mathbb{1}g'(\lambda')$, let $g'_i(\lambda') = g_i(\lambda')G(\lambda')/\lambda'$ be the not necessarily scaled $N$-path. Let $\lambda'$ satisfy the non-scaled equation

$$\sum_{i=1}^{n} \min\left\{g'_i(\lambda'), c'_i\right\} = E'$$

for claims $c'$ and endowment $E'$ and let $x_i = \min\left\{g'_i(\lambda'), c'_i\right\}$ be the corresponding award.

**Proposition 1** *Suppose $\psi'(\lambda') = W'$ where $W'$ is a number independent of $\lambda'$. Then $E = E'/W'$ and $c = c'/W'$ turns the non-scaled equation into a scaled equation, and vice versa, while $x = x'$. If $g'_i(\lambda') = w'_i\lambda'$ (weighted gains) then $g_i(\lambda') = w'_i\lambda'/W'$ and $W' = \mathbb{1}w'$. First, the weights $w'_i$ can be scaled to $w_i = w'_i/\mathbb{1}w'$ with $\lambda = \lambda'$, but there is also a second scaling: again (necessarily) $w_i = w'_i/\mathbb{1}w'$ but $E = E'$ and $c = c'$ with $\lambda = \lambda'\mathbb{1}w'$.*

[6] If the weights are individual endowments, $\mathbb{1}w = E$, then the mechanism is the proportional uniform rule [41]; in brief, participants who own an endowment greater than their peak, the suppliers, have a surplus, which is distributed over demanders in proportion to their individual endowment; and repeat this process.

[7] Moulin [30] introduced fixed-path rationing and proved that it is the only mechanism satisfying efficiency, resource-monotonicity, and strategy-proofness, with a 'straightforward' proof given by Ehlers [14].

**Proof**. Substituting $g'_i(\lambda) = \psi'(\lambda) g_i(\lambda)$ in the non-scaled equation yields

$$\sum_{i=1}^{n} \psi'(\lambda') \min\left\{ g_i(\lambda'), \frac{c'_i}{\psi'(\lambda')} \right\} = E'$$

which shows how it can become an equation for fixed-path distribution:

$$\sum_{i=1}^{n} \min\left\{ g_i(\lambda'), \frac{c'_i}{W'} \right\} = \frac{E'}{W'}$$

For weighted gains, this equation gives the first scaling, while

$$\sum_{i=1}^{n} \min\left\{ w_i \lambda' W', c'_i \right\} = E'$$

yields the second scaling. ■

This last equation of course is trivial.

**Example 4** *The uniform rule has $w'_i = 1$ for all $i$ so $W' = n$ (so non-scaled weighted gains) and $w_i = 1/n$. Further, $g'_i(\lambda') = \lambda' w'_i = \lambda'$ is scaled as $g_i(\lambda') = \lambda'/n$. Let $n = 3$ and $c' = (1, 2, 3)$. For $E' = 5$ one finds $x' = (1, 2, 2)$ for $\lambda' = 2$. Scaling: $w = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Scaling 2: $c = c'$ and $E = E'$ so $\lambda = 6$ yields $x = x'$. Scaling 1: $c = (\frac{1}{3}, \frac{2}{3}, 1)$ and $E = \frac{5}{3}$ so $\lambda = \lambda'$ yields $x = x'$.*

**Example 5 (Proportional Uniform Rule)** *Consider non-scaled weighted gains, that is, let $g(\lambda') = w'_i \lambda'$ and $\mathbb{1} g(\lambda') = E' \lambda'$ so $\mathbb{1} w' = E'$. The $w'_i$ are individual endowments (possessions) and the $c'_i$ are claims, where $c'_i > w'_i$ for demanders and $c'_i < w'_i$ for suppliers. This weighted gains is called the* proportional uniform rule *[41]. Let $w := w'/E'$. First scaling: $E = 1$ and $c = c'/E'$. Second scaling: $E = E'$ and $c = c'$.*

## A.2 Minology

The following inequalities for minima are needed in the following section about merging and splitting, or are useful for improvements. As is easily seen,

$$\max\{x, y\} \leq z \Leftrightarrow (x \leq z \wedge y \leq z) \Rightarrow (x \leq z \vee y \leq z) \Leftrightarrow \min\{x, y\} \leq z \quad (3)$$

and

$$\min\{x, y\} \geq z \Leftrightarrow (x \geq z \wedge y \geq z) \Rightarrow (x \geq z \vee y \geq z) \Leftrightarrow \max\{x, y\} \geq z \quad (4)$$

for any $x$, $y$, and $z$. Boundary cases are more convoluted, for example

$$\begin{aligned} \min\{x, y\} = z \quad & \Leftrightarrow (z = x \leq y \vee z = y < x) \\ & \Leftrightarrow (z = x < y \vee z = y < x \vee x = y = z) \end{aligned} \quad (5)$$

follows from either equation. Potentially of interest:

$$\begin{aligned} x + y \quad & = \max\{x, y\} + \min\{x, y\} \\ |x - y| \quad & = \max\{x, y\} - \min\{x, y\} \end{aligned}$$

The triangular inequality $|x + y| \leq |x| + |y|$ only gives trivial results. While dealing with positive $x$ and $y$ only, the reverse triangular inequality $|x - y| \geq \big||x| - |y|\big|$ is not helpful either.

The following is needed to bundle pairs of claims and weights in fixed-path distribution as a minimum of joint claims and weights.

For merging claims in the fixed-path distribution, $x_i = \lambda w_i$ and $y_i = c_i$. A comparison with min is needed. For the final result, which involves $\mu$, deciding whether $x_i \leq y_i$ would depend on $\lambda$, of which the computation is possible but to be avoided in practice. The conditions for equality are not needed here.

**Proposition 2** *For $i$ in some finite set $K$ of cardinality $k = |K|$ let $x_i$ and $y_i$ be real-valued numbers. Then*

$$\min\{kx^\dagger, ky^\dagger\} \leq \sum_i \min\{x_i, y_i\} \leq \min\{\mathbb{1}x, \mathbb{1}y\}$$

*for $x^\dagger := \min\{x_i \mid i \in K\}$, which defines $y^\dagger$ similarly. Left-hand equality iff $x = y$ and $x_i = x_j$ for all $i$ and $j$. Right-hand equality iff $x \leq y$ or $x > y$ (coordinate-wise inequalities).*

**Proof**. Let $i$ range over $K$ in the summations and definitions of sets. Let $\sigma := \sum_i \min\{x_i, y_i\}$. Basis: $\sigma = \sum_{i:x_i \leq y_i} x_i + \sum_{i:x_i > y_i} y_i$.

Right-hand inequality: Basis yields $\sigma \leq \sum_i y_i = \mathbb{1}y$ and $\sigma \leq \sum_i x_i = \mathbb{1}x$. Invoke Equation 4 to prove the inequality. Equality iff $\sigma$ contains one sum: $\sigma = \mathbb{1}x \leq \mathbb{1}y$ or $\sigma = \mathbb{1}y < \mathbb{1}x$. Resort to Equation 5.

Left-hand inequality: Introduce $x^* := \min\{x_i \mid x_i \leq y_i\}$ and $y^\bullet := \min\{y_i \mid x_i > y_i\}$. The basis yields[8] $\sigma \geq \tau$ for $\tau := \sum_{i:x_i \leq y_i} x^* + \sum_{i:x_i > y_i} y^\bullet$, which are sums of constants. (Equality $\sigma = \tau$ iff $x = y$, in which case $\sigma = \mathbb{1}x$.) Distinguish $x^* \leq y^\bullet$ and $x^* \geq y^\bullet$. So $\tau \geq kx^*$ or $\tau \geq ky^\bullet$. Equation 3: $\tau \geq \min\{kx^*, ky^\bullet\}$. Finally, $x^* \geq x^\dagger$ and $y^\bullet \geq y^\dagger$. So $\tau \geq \min\{kx^\dagger, ky^\dagger\}$, which proves the inequality. Equality iff $\tau = kx^\dagger \leq ky^\dagger$ or $\tau = ky^\dagger < kx^\dagger$ according to Equation 5. As $x = y$ is required already for $\sigma = \tau$, a necessary and sufficient condition is $x = y$ and $\mathbb{1}x = \sigma = kx^\dagger$, and the latter tells that all $x_i$ are the same. ■

## A.3 Claims Merging and Splitting

The following supports Question 7 on page 11. The point of departure is weighted gains but some other $N$-path of fixed-path distribution may be necessary, some 'weighted' proportional uniform rule, or a need-based mechanism instead of one based on single-peaked preferences.

The purpose of merging claims of a group $Q$ of people is to reduce administration. For example, nobody applying for a bread should need to apply for grain, water and so on, which would have to be distributed separately. Can participants $Q$ merge claims and effort indices (weights)

[8]The right-hand side is $\geq \left(\sum_{i:x_i \leq y_i} 1\right) \min_{i:x_i \leq y_i} x_i + \left(\sum_{i:x_i > y_i} 1\right) \min_{i:x_i > y_i} y_i$ but this can not sensibly be simplified.

such that claims and weights of others need not be adapted and that the joint award probably approximately equals the total of the unmerged awards?

The uniform rule (constrained equal awards for the case of scarcity) is not manipulable by merging [28, Th. 1ii] or merging-proof [39, p. 54]. The proportional rule is the only such rule obeying certain additional requirements [35, p. 345].

Let the joint claim be $\Gamma$ and $V$ the joint weight. Define $M = N \setminus Q$ as the rest. The original equation

$$\sum_{i\in Q} \min\{w_i\lambda, c_i\} + \sum_{i\in M} \min\{w_i\lambda, c_i\} = E$$

when solved for $\lambda$ yields awards $x_i = \min\{w_i\lambda, c_i\}$. So $\mathbb{1}x|_Q + \mathbb{1}x|_M = E$. The merged equation is

$$\min\{V\mu, \Gamma\} + \sum_{i\in M} \min\{w_i\mu, c_i\} = E$$

where $c|_M$ and $E$ are unaltered for simplicity. When solved for $\mu$ the awards are $Y = \min\{V\mu, \Gamma\}$ and $y_i = \min\{w_i\mu, c_i\}$ for $i$ in $M$. So $Y + \mathbb{1}y|_M = E$.

Is there a combination of weights $V$ and combination of claims $\Gamma$ such that the award $Y$ after after merging probably is approximately equal to the total return $\mathbb{1}x|_Q$ without merging? In other words, purpose is to have $x|_M \approx y|_M$. (The distribution of $Y$ amongst $Q$ is of no concern yet but if it turns out to deviate from $\mathbb{1}c|_Q$ then a fixed-path rationing will be most natural, though that requires an additional computation.)

Weighted gains meets consistency, upper and lower composition and scale invariance [31, p. 658] but this is not directly related.

Let $c|_Q^\dagger := \min\{c_i \mid i \in Q\}$ and $w|_Q^\dagger := \min\{w_i \mid i \in Q\}$, as in Proposition 2.

**Example 6 (Classic uniform rule)** *Claims $c = (1, 2, 3)$ and endowment $E = 4$. So*

$$\min\{\lambda, 1\} + \min\{\lambda, 2\} + \min\{\lambda, 3\} = 4$$

*yields $\lambda = 3/2$ whence $x = (1, \frac{3}{2}, \frac{3}{2})$. Coalition $Q = \{1, 3\}$. Merge by adding the claims: $\Gamma = \mathbb{1}c|_Q = c_1 + c_3 = 4$. The weights are $w = (1, 1, 1)$ and to retain the form of the uniform rule, let $V = 1$. Merging gives*

$$\min\{\mu, 4\} + \min\{\mu, 2\} = 4$$

*so $\mu = 2$. So 'merging-proof': $Y = 2 < 2\frac{1}{2} = x_1 + x_2$ and $y_2 = 2 > x_2 = \frac{3}{2}$.*

For the following propositions, such a mixture of claims addition and weights minimum is not considered.

**Proposition 3 (Bundling by Summation)** *Suppose $\mathbb{1}c > E$. If $\Gamma = \mathbb{1}c|_Q$ and $V = \mathbb{1}w|_Q$ then the joint award $Y = \min\{\mu\mathbb{1}w|_Q, \mathbb{1}c|_Q\}$ satisfies $Y \geq \mathbb{1}x|_Q$ so bundling by summation of claims and weights is not strategy-proof or yields the same outcome.*

**Proof**. Let $f(\nu) := \min\{\nu\mathbb{1}w|_Q, \mathbb{1}c|_Q\} + \sum_{i\in M}\min\{\nu w_i, c_i\}$ for any $\nu$. Let $Z := \min\{\lambda\mathbb{1}w|_Q, \mathbb{1}c|_Q\}$. As $Z \geq \sum_{i\in Q}\min\{\lambda w_i, c_i\} = \mathbb{1}x|_Q$ (second inequality in Proposition 2) one finds

$$Z + \mathbb{1}x|_M \geq \mathbb{1}x|_Q + \mathbb{1}x|_M = E = Y + \mathbb{1}y|_M$$

so $f(\lambda) \geq f(\mu)$. On the range of $\mu$, which includes 0, the function $f$ is strictly increasing, for otherwise $\mathbb{1}c = E$. So $\lambda \geq \mu$. Therefore, $\min\{\lambda w_i, c_i\} \geq \min\{\mu w_i, c_i\}$ for all $i$ in $M$. In other words, $x|_M \geq y|_M$, that is, the participants not in the coalition may be worse off. So $\mathbb{1}x|_M \geq \mathbb{1}y|_M$ and $Y = E - \mathbb{1}y|_M \geq E - \mathbb{1}x|_M = \mathbb{1}x|_Q$: the coalition may be better off. ■

An example is given after the next proposition.

**Proposition 4 (Bundling by Minimisation)** *Suppose $\mathbb{1}c > E$. Let $q := |Q|$. If $\Gamma = qc|_Q^\dagger$ and $V = qw|_Q^\dagger$ then the joint award $Y = \min\{\mu qw|_Q^\dagger, qc|_Q^\dagger\}$ satisfies $Y \leq \mathbb{1}x|_Q$ on the condition that $qc|_Q^\dagger + \mathbb{1}c|_M > E$, so after bundling, $E$ still does not suffice for all. In that case, bundling by summation of minimum claims and weights is strategy-proof.*

**Proof**. Let $g(\nu) := \min\{\nu qw|_Q^\dagger, qc|_Q^\dagger\} + \sum_{i\in M}\min\{\nu w_i, c_i\}$ for any $\nu$. Let $\Omega := \min\{\lambda qw|_Q^\dagger, qc|_Q^\dagger\}$. As $\Omega \leq \sum_{i\in Q}\min\{\lambda w_i, c_i\} = \mathbb{1}x|_Q$ (first inequality in Proposition 2) we find

$$\Omega + \mathbb{1}x|_M \leq \mathbb{1}x|_Q + \mathbb{1}x|_M = E = Y + \mathbb{1}y|_M$$

so $g(\lambda) \leq g(\mu)$. Some $\mu$ is to be found in the range $\{\nu \mid g(\nu) \leq E\}$ so that $g(\mu) = E$. In that range, $g$ is strictly increasing (depends on $\nu$) because otherwise $g(\nu) = qc|_Q^\dagger + \mathbb{1}c|_M > E$. For $\lambda$ in the range, $\lambda \leq \mu$ as $g$ is strictly increasing. Now let $\lambda$ be outside the range, so $g(\lambda) > E$. Would $\lambda > \mu$ then $g(\lambda) \geq g(\mu)$ because $g$ is increasing or constant. As $g(\lambda) \leq g(\mu)$ the only remaining case is $g(\mu) = g(\lambda)$ but then $g(\mu) > E$, a contradiction. So again, $\lambda \leq \mu$. As before, $x|_M \leq y|_M$, so $\mathbb{1}x|_M \leq \mathbb{1}y|_M$, so $Y = E - \mathbb{1}y|_M \leq E - \mathbb{1}x|_M = \mathbb{1}x|_Q$. ■

The case $qc|_Q^\dagger + \mathbb{1}c|_M < E$ yields an equation $\mu$ in terms of max and requires further research.

**Example 7 (scaled Uniform Rule)** *Claims $c = (1, 2, 3)$ and endowment $E = 4$. Scaling 2 gives a scaled uniform rule: $w = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. So $\lambda = 9/2$ and $x = (1, \frac{3}{2}, \frac{3}{2})$ as for the classic uniform rule.*

*Bundling by summation, $\Gamma = 4$ and $V = \frac{2}{3}$, gives the merged equation*

$$\min\left\{\frac{2}{3}\mu, 4\right\} + \min\left\{\frac{1}{3}\mu, 2\right\} = 4$$

*So $\mu = 4$ whence $Y = 2\frac{2}{3} > x_1 + x_3 = 2\frac{1}{2}$ and $y_2 = \frac{8}{6} < x_2 = \frac{9}{6}$. So not 'merging-proof'.*

*Bundling by minimisation: $\Gamma = 2\min\{1, 3\} = 2$ and $V = 2\min\{\frac{1}{3}, \frac{1}{3}\} = \frac{2}{3}$ so still scaled. The merged equation*

$$\min\left\{\frac{2}{3}\mu, 2\right\} + \min\left\{\frac{1}{3}\mu, 2\right\} = 4$$

*can still use* min *and yields* $\mu \geq 6$ *so* $Y = 2 < 2\frac{1}{2}$ *and* $y_2 = 2 > 1\frac{3}{2}$ *so 'merging-proof'.*

Now for different weights.

**Example 8 (Weighted gains)** *Claims* $c = (1, 2, 3)$*, endowment* $E = 5$*, and weights* $w = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$*. So*

$$\min\{\frac{1}{2}\lambda, 1\} + \min\{\frac{1}{3}\lambda, 2\} + \min\{\frac{1}{6}\lambda, 3\} = 5$$

*and the solution* $\lambda = 5$ *yields* $x = (1, 2, 2)$.

*Coalition* $Q = \{1, 3\}$ *and bundling by summation yields* $\Gamma = c_1 + c_3 = 4$ *and* $V = w_1 + w_3 = \frac{2}{3}$*. So*

$$\min\{\frac{2}{3}\mu, 4\} + \min\{\frac{1}{3}\mu, 2\} = 5$$

*whence* $\mu = 5$ *and* $Y = \frac{10}{3} > 3 = x_1 + x_3$ *and* $y_2 = \frac{5}{3} < 2 = x_2$*. So not 'merging-proof'.*

*Bundling by minimisation:* $\Gamma = 2\min\{1, 3\} = 2$ *and* $V = 2\min\{\frac{1}{2}, \frac{1}{6}\} = \frac{1}{3}$ *so not scaled as* $\mathbb{1}w = \frac{2}{3} < 1$*. Proposition 4 does not apply because the merged equation*

$$\min\{\frac{1}{3}\mu, 2\} + \min\{\frac{1}{3}\mu, 2\} = 5$$

*does not have* $2 + 2 = \Gamma + c_2 > E = 5$*. If rephrased with* max*, then* $\mu = \frac{15}{2}$ *and* $Y = y_2 = 2\frac{1}{2}$ *so 'merging-proof'.*

The remainder of this section is an investigation into the extent to which there can be an expression for the joint claim $\Gamma$ and the joint weight $V$ of the coalition $Q$ such that $Y = \mathbb{1}x|_Q$, which is to say, such that $\min\{\mu V, \Gamma\} = \sum_{i \in Q} \min\{\lambda w_i, c_i\}$. This is generally impossible because the only rule not allowing advantageous merging or splitting is the proportional rule [38, pp. 286-287]. So, the following is a first step towards finding a joint expression that allows to approximate the desired equalities.

If strict equalities were possible, it would be unfair if the awards of $M$ (the rest) would change, so that some would be better off and others not. Therefore, investigate the possibility of the existence of $\Gamma$ and $V$ such that $y|_M = x|_M$. Assume $\mathbb{1}c > E$ and $\Gamma + \mathbb{1}c|_M > E$. This would imply

$$Y = \min\{\mu V, \Gamma\} = E - \mathbb{1}y|_M = E - \mathbb{1}x|_M = \sum_{i \in Q} \min\{\lambda w_i, c_i\} = \mathbb{1}x|_Q$$

when no scaling. For any $\kappa$ define $\alpha(\kappa) := \sum_{i \in M} \min\{\kappa w_i, c_i\}$. So

$$\alpha(\lambda) + \sum_{i \in Q} \min\{\lambda w_i, c_i\} = E$$

is to be merged as $\alpha(\mu) + \min\{\mu V, \Gamma\} = E$. Generally, $\alpha(\lambda) = \mathbb{1}x|_M = \mathbb{1}y|_M = \alpha(\mu)$.

Case $\lambda w|_M \geq c|_M$. So $x|_M = c|_M$ whence $y|_M = c|_M \leq \mu w|_M$. Let $j$ in $Q$ be such that $\lambda w_j < c_j$. (Such $j$ exists.) So:

$$Y = \min\{\mu V, \Gamma\} = \sum_{i \in Q} \min\{\lambda w_i, c_i\} < \mathbb{1}c|_Q$$

Notably, $\sum_{i\in Q} \min\{\lambda w_i, c_i\} < \min\{\lambda \mathbb{1} w|_Q, c|_Q\}$ because $\lambda w|_Q \ngeq c|_Q$ and $\lambda w|_Q \nleq c|_Q$ as in Proposition. Subcase $\mu V < \Gamma$ gives $Y = \mu V$ and from $Y = E - \mathbb{1}y|_M = E - \mathbb{1}c|_M$ follows $\boxed{\mu V = E - \mathbb{1}c|_M}$ and $\mu(V + \mathbb{1}w|_M) \geq E$ because $c|_M \leq \mu w|_M$. Subcase $\mu V \geq \Gamma$ yields $\boxed{\Gamma < \mathbb{1}c|_Q}$, which is not very informative because it already follows from $Y = E - \mathbb{1}y|_M = E - \mathbb{1}c|_M < \mathbb{1}c|_Q$.

Case $\lambda w|_M \ngeq c|_M$. Let $i$ in $M$ obey $\lambda w_i < c_i$. So $\alpha(\lambda) < \mathbb{1}c|_M$. Therefore, $\min\{\mu V, \Gamma\} = Y = E - \alpha(\mu) = E - \alpha(\lambda) > E - \mathbb{1}c|_M$ So $\boxed{\mu V > E - \mathbb{1}c|_M}$ and $\boxed{\Gamma > E - \mathbb{1}c|_M}$. Also $\min\{\mu w_i, c_i\} = y_i = x_i = \lambda w_i$, so $\mu w_i \leq c_i$, from which $\lambda = \mu$. So

$$\min\{\lambda V, \Gamma\} = \sum_{i \in Q} \min\{\lambda w_i, c_i\} \leq \min\{\lambda \mathbb{1} w|_Q, \mathbb{1}c|_Q\}$$

where the last inequality follows from Proposition. So $V \leq \mathbb{1}w|_Q$ in this case.

The boxes for $\Gamma$ are compatible as $\mathbb{1}c > E$ but those for $\mu V$ are not, so a general bundling that does not affect unbundled claims seems impossible. This analysis only suggests that a bundling that least affects other rewards obeys $\max\{E - \mathbb{1}c|_M, 0\} < \Gamma < \mathbb{1}c|_Q$ and $0 < V \leq \mathbb{1}w|_Q$ where the claims $c|_M$ are known to a computer program but not to the coalition $Q$.

# B Priorities

Participants can first be sorted on priorities and for each priority, receive goods according to the fixed-path rationing; this does not compromise strategy-proofness [30].

If the combination of consistency and resource-monotonicity (a condition for fixed-path rationing) is replaced with replacement monotonicity, then the only option is *sequential allotment*, where the $N$-path is not parametrized but changes more freely in the course of the computation. This allows to assign priorities based on previous stages of the computation [5]. Sequential allocation also allows discrete allotment, see [3, App. 2].

# C Generalised Reciprocity

*Generalised reciprocity* or *generalised exchange* is the remuneration of a benefactor not necessarily by the beneficiary [17, p. 421 note 21]. The term 'mutual aid' is also used but the word 'mutual' may be conceived as *strict reciprocity*, that is, where the beneficiary remunerates.

Reciprocity: 'If I need a good or service from a person then I should reciprocate to that person the donation of the good or the rendering of the service: either I have a debt to redeem or I pay in advance.'

Generalised reciprocity: 'If I need a good or service, I can be quite certain that someone from my group will provide it but I should reciprocate to the group: either I have a debt to redeem or I pay in advance.' One example is money: 'I work for someone (and get money) and the *good or service* is obtained not necessarily from that person (by paying the money).' Another example is fixed-path distribution: 'I work for someone (and get an effort index) and the *good* is obtained not necessarily from that person (by letting the effort weigh in during fixed-path distribution). This covers the reversed situation: if I need labour, I am in the same situation as that other person and will register the effort.'

**Example 9** *In a number of European countries, Eurotransplant determines who receives an organ for transplantation for most kinds of donors. For livers, reciprocity holds: If a liver crosses a border then the recipient country returns a liver to the donor country at the earliest occasion. (These 'liver obligations' are geographically connected in the obvious way but this does not prevent high transport cost.) For kidneys, reciprocity is generalised: candidate kidney recipients are ranked higher for a kidney from abroad in proportion to the net kidney export of their country, irrespective of the country that received the kidney [15].*

If the Latin 'do ut des' (I give so that you give) is conceived as a succinct paraphrasing of strict reciprocity, then generalised reciprocity would become 'do ut det': I give (to you) so that he or she gives.

# D Inspection-Based Mechanisms

Agent D (the declarer) owns an object which has a value $\theta$ in $\mathbb{R}_{\geq 0}$ that initially is known only to D. (So, D has no preference for a particular $\theta$.) Agent C (the claimant or perhaps challenger) may ask D to reveal $\theta$, that is, force an inspection of $\theta$, but also ask D to just report $\theta$. The reported $\theta$ is $\hat{\theta}$. Agent C may accept the declaration or buy the object. Let the transfer function $t$ map $\theta$ to some real-valued number, the payment. Suppose $t$ is a strictly increasing function and that $t(\theta) \leq \theta$ for all $\theta$. The following utilities are paid off:

$$U_{\mathrm{D}}(\theta) = \begin{cases} \theta - t(\hat{\theta}) & \text{if accepted} \\ \hat{\theta} - \theta & \text{if bought} \end{cases} \qquad U_{\mathrm{C}}(\theta) = \begin{cases} t(\hat{\theta}) & \text{if accepted} \\ \theta - \hat{\theta} & \text{if bought} \end{cases}$$

An example is the Liturgeia, where a citizen C of ancient Athens suspects a cocitizen D to be rich enough to contribute to the fleet. If C suspects that $\hat{\theta} < \theta$ then C may decide to buy and get the true value $\theta$ [21]. Otherwise, D will have to contribute $t(\hat{\theta})$ which for simplicity of the model is a payment to C. Another example is levying a tax, such as $t(x) = x/100$, in various settings [18].

Suppose that D needs a positive outcome. The outcome would be negative if $\hat{\theta} < \theta$ and the challenger decides to buy. Therefore $\hat{\theta} \geq \theta$ but then $t(\hat{\theta})$ should be minimized. So $\hat{\theta} = \theta$.

Te mechanism hinges on the possibility that the physical reality is disclosed [22]. When D does not need a positive outcome, the mechanism is not truth-telling [18]. By the Bayesian revelation principle [25, p. 291], a mechanism not based on preferences exists that does not require inspections but discourages to misreport one's type without inspection. The question is how that mechanism looks like. The following might be a search in the wrong direction.

# E Need-Based Cake Division

Following are two unsatisfactory procedures to elicit a need instead of a want of two agents for a piece of a homogeneous cake. The cake may be divisible at arbitrary or predetermined points (be 'continuous' or 'discrete') and the selection of the pieces is random. In the first procedure, one agent cuts the cake but not much is learned. In the second procedure, both agents cut at the same time; in the hypothetical case that the agents avoid mutual destruction, this reveals the needs.

## E.1 Cake Cutting with Random Selection

First, suppose agent 1 needs at least a piece $\theta$ of a cake of size $E$ because otherwise, he or she will die. Futher, agent 1 wants as large a piece as possible. If agent 2 cuts based on inspection, then agent will always choose $\hat{\theta} = E/2$, so this does not tell anything about $\theta$. Now suppose both pieces of cake are concealed in identical boxes. Agent 2 choses a box (without weighing it) with equal probability and the piece $x$ therein is given to agent 1. The idea is that maximizing the utility is bounded and this bound depends on the need as the selection may turn out to be the other piece of cake. If $\theta > E/2$ then agent 1 will put the whole cake in a box and hope for the best. If $\theta \leq E/2$ then suppose agent 1 splits the cake at $\hat{\theta}$ where $\theta \leq \hat{\theta} \leq E - \theta$ as any $\hat{\theta}$ outside that range poses a risk for getting less than needed. The expected size is $\frac{1}{2}\hat{\theta} + \frac{1}{2}(E - \hat{\theta}) = E/2$, which is independent of $\hat{\theta}$. So, $\hat{\theta}$ is uniformly distributed in this interval and all one can conclude is that $\theta \leq \min\{\hat{\theta}, E - \hat{\theta}\}$.

## E.2 Sealed Cake Bids with Random Selection

Second, let agent 2 cut at the same time as agent 1, or rather, both make a sealed bid $\hat{\theta}_i$ where $\hat{\theta}_i \geq \theta_i > 0$ for $i = 1$ and $i = 2$. There is a random selection of the agent who gets the piece bid for; if $x_i < \theta_i$ then agent $i$ dies, in which case $x_i$ is given to agent $3 - i$ (the other). So, the award is

$$x = (x_1, x_2) := \begin{cases} \begin{cases} (\hat{\theta}_1, E - \hat{\theta}_1) & \text{if } E - \hat{\theta}_1 \geq \theta_2 \\ (E, 0) & \text{otherwise} \end{cases} & \text{probability } p \\ \begin{cases} (E - \hat{\theta}_2, \hat{\theta}_2) & \text{if } E - \hat{\theta}_2 \geq \theta_1 \\ (0, E) & \text{otherwise} \end{cases} & \text{probability } 1 - p \end{cases}$$

where $0 < p < 1$. If $\hat{\theta}_1 + \hat{\theta}_2 \leq E$ then the remainder $E - \hat{\theta}_1 - \hat{\theta}_2$ is given randomly to an agent but it could also be distributed in some convenient way, for example, split equally, at least for a 'continuous' cake.

**Example 10** *Agent 2 can choose a very large piece and have a chance to kill agent 1. Suppose $\theta_1 + \hat{\theta}_2 > E$. So $\hat{\theta}_1 + \hat{\theta}_2 > E$. If it turns out that $x_1 = E - \hat{\theta}_2$ then $x_1 < \theta_1$ so agent 1 dies. Suppose also $\hat{\theta}_1 + \theta_2 \le E$. If $x_2 = E - \hat{\theta}_1$ happens, then $x_2 \ge \theta_2$ so agent 2 survives. For example, $E = 100$, $\theta = (1, 1)$ and $\hat{\theta} = (2, 100)$ yields $x = (2, 98)$ or $x = (0, 100)$. If $p = 1/2$ then agent 1 still expects to get* 1.

A dominant strategy equilibrium is as set of agent's responses each of which is best whatever the other agent does [25, pp. 5, 283].

**Proposition 5 (Dominant Strategy Equilibrium)** *The moves $\hat{\theta} = (E, E)$ constitute a dominant strategy equilibrium.*

**Proof**. Agent 2 reasons as follows. If $\hat{\theta}_1 \le E - \theta_2$ then my bid should have been $\hat{\theta}_2 = E$ because the outcome $x = (\hat{\theta}_1, E - \hat{\theta}_1)$ saves me $(E - \hat{\theta}_1 \ge \theta_2)$ and $x = (0, E)$ is optimal. If $\hat{\theta}_1 > E - \theta_2$ (for example, agent 1 makes a grossly exagerated bid $\hat{\theta}_1 = E$) then my bid should have been $\hat{\theta}_2 = E$ too: the awards are $x = (E, 0)$ or $x = (0, E)$ so if I survive, I get the whole cake. Agent 1 reasons similarly, or knows that agent 2 reasons this way, so will also choose $\hat{\theta}_1 = E$. ■

For $i = 1$ and $i = 2$ let $\hat{\theta}_i(\omega_1)$ be a random variable depending on $\omega_i$ from some sample space $\Omega_i$ of a probability space $\mathbb{P}_i$ not depending on $\hat{\theta}_{3-i}$, the opponent of $i$. Let $X_i(\omega_i) := E - \hat{\theta}_{3-i}(\omega_i)$. The purpose of the game (truth telling) is expressed by the following proposition, which however is void because it demands that $\mathbb{P}_i(X_i < y)$ be strictly increasing in $y$. However, consider $i = 1$. In the dominant strategy equilibrium $\hat{\theta} = (E, E)$ one has $\mathbb{P}_1(X_1 < \hat{\theta}_1) = \mathbb{P}_1(E - \hat{\theta}_2 < \hat{\theta}_1) = \mathbb{P}_1(0 < \hat{\theta}_1) = 1$ and thus, constant in $\hat{\theta}_1$.

**Proposition 6 (Strategy-proofness)** *If $\mathbb{P}_i(X_i < y)$ is strictly increasing in $y$ then the procedure elicits the true needs.*

**Proof**. Concentrate on agent 1 as agent 2 is treated similarly. So $X_1 = E - \hat{\theta}_2$. First, suppose $\hat{\theta}_1 + \hat{\theta}_2 \le E$. Then $x_1 = \hat{\theta}_1 \ge \theta_1$ and $x_2 = E - \hat{\theta}_1 \ge \hat{\theta}_2 \ge \theta_2$ so both agents get their needs. Next, consider the only risky circumstance, $\hat{\theta}_1 + \hat{\theta}_2 > E$, which could also turn out to be the case. It may happen that $E - \hat{\theta}_2 < \theta_1$, which is fatal for agent 1. The only way agent 1 can influence the event $X_1 < \theta_1$ is by diminishing the probability $\mathbb{P}_2(X_1 < \hat{\theta}_1)$, that is, of $\hat{\theta}_1 + \hat{\theta}_2 > E$ happening. This probability is minimised by choosing the least $\hat{\theta}_1$. So $\hat{\theta}_1 = \theta_1$. ■

The question is how $\hat{\theta} = (E, E)$ can be avoided or whether a variation of this game allows truth-telling. Perhaps the following section provides a clue.

# F Planning Game

The estimation of effort, reported as $\hat{f}$, and the actual effort, registered as $\hat{e}$, basically are subject to the following game, where $\mu$ in the role of $\hat{f}$ and $c$ in the role of $\hat{e}$.

There are participants 1 and 2. An entity (for instance, total effort) of maximum size $E$ is to be determined. Each participant $i$ reports $\mu_i$, the size of his or her part of the entity. The total size $E := \mu_1 + \mu_2$ is not public, so $i$ cannot derive $\mu_j$, where $j := 3 - i$ is the other. Once the entity is determined, the 'real' parts $c_i$ are reported (possibly by $j$ or nature). The outcomes $x_i$ are such that any surplus $\mu_j - c_j > 0$ is given to $i$ when $i$ has a deficit ($c_i - \mu_i > 0$), that is:

$$x_i := \begin{cases} \mu_i + \min\{0, \mu_j - c_j\} & \text{if } c_i > \mu_i \\ c_i & \text{if } c_i \leq \mu_i \end{cases}$$

The person $i$ having the largest $x_i$ is the winner. (In the economy of the main text, the amount of goods is greatest if the accounted effort $x_i$ is greatest, at least if both required equal amounts.) If $i$ underestimated ($c_i > \mu_i$) then $i$ may see the surplus go to $j$ and if $i$ overestimated ($c_i \leq \mu_i$) then $i$ only gets $c_i$, the low value. To be determined is whether the choice of $c$ yields any strategy-proof mechanism, that is, where estimating $\mu = c$ is encouraged. Also, to be investigated is whether the 'punishment' can be replaced with a mechanism that suppresses collusion when making the estimates.

To relate to bankruptcy with claims guarantees, Appendix G, let $\mathbb{1}x := x_1 + x_2$ and similarly for other variables. So:

$$\begin{array}{llllll} \mathbb{1}x < \mathbb{1}\mu < \mathbb{1}c & \text{if} & c_1 > \mu_1 & \text{and} & c_2 > \mu_2 \\ \mathbb{1}x \leq \mathbb{1}\mu \text{ and } \mathbb{1}x < \mathbb{1}c & \text{if} & c_1 > \mu_1 & \text{and} & c_2 \leq \mu_2 \\ \text{same} & \text{if} & c_1 \leq \mu_1 & \text{and} & c_2 > \mu_2 \\ \mathbb{1}x = \mathbb{1}c \leq \mathbb{1}\mu & \text{if} & c_1 \leq \mu_1 & \text{and} & c_2 \leq \mu_2 \end{array}$$

So $\mathbb{1}x \leq \max\{\mathbb{1}c, \mathbb{1}\mu\}$.

For 3 or more players, the surplus may be distributed using the functions $\psi$ and $\rho$ in Appendix A.1 and Appendix G.6, respectively, possibly allocating most of the surplus from $j$ still to $i$.

# G Bankruptcy with Claims Guarantees

In essence, a bank employs claims guarantees if upon bankruptcy, claims below the guarantee are awarded to the claimant and for claims greater than the guarantee, the award is also greater than the guarantee. There are several ways to let the awards be partially proportional to the claim.

## G.1 Notations and Conventions

As usual, 'iff' means 'if and only if'. The identity function is id. Unless stated otherwise, numbers are in $\mathbb{R}_{\geq 0}$, that is, real-valued and non-negative. Consider a set $X$ and suppose $A$ is subset of $X$. Let $A^\neg := X \setminus A$ be the *complement* of $A$ with respect to $X$. Let $Y$ be another set, possibly equal to $X$. Consider any function $F$ from $X$ to $Y$ as a subset of $X \times Y$. The function value of some $x$ from $X$ is denoted $F(x)$ or $F_x$ but in proofs, it may yet be abbreviated to just $F$ if $x$ is fixed. Let $F|_A := F \cap (A \times Y) = \Big\{\big(a, F(a)\big) \mid a \in A\Big\}$, the *restriction* of $F$ to $A$. Let $N := \{1, \ldots, n\}$ for some natural number $n$. Consider a function

$f : N \to \mathbb{R}$, so $f$ is a subset of $N \times \mathbb{R}$. Its projection on $\mathbb{R}$ is the *$f$-vector* or *$f$-profile* $f[N] = (f_1, \dots, f_n)$. Let $M$ be a subset of $N$. So $f|_M := \{(j, f_j) \mid j \in M\}$. For any $g : M \to \mathbb{R}$ let $\mathbb{1}g := \sum_{j \in M} g_j$ where $\mathbb{1} := (1, 1, \dots, 1)$ is a row of the appropriate dimension, $|M|$ in this case, and $g$ is thought of as a column, so $\mathbb{1}g$ is an inner product. Therefore, $\mathbb{1}f|_M = \sum_{j \in M} f_j$.

For $n$-tuples $u$ and $v$ let $u \le v$ denote $u_i \le v_i$ for all $i$ in $N$, and similarly $u < v$. So $u \not< v$ denotes that $u_i < v_i$ not for all $i$, that is, there is an $i$ in $N$ such that $u_i \ge v_i$. Therefore, $\mathbb{1}u < \mathbb{1}v$ implies that $u_i < v_i$ for some $i$ (for, suppose the contrary).

Let $\preceq$ be another notation for $\le$, the component-wise comparison. Further, $\prec$ is defined as the asymmetric part of $\preceq$, that is, $u \prec v$ means $u \le v$ and $u \not\ge v$, which is to say that $u \le v$ and $u_i < v_i$ for some $i$. Therefore, $u \not\prec v$ is $(u \le v \Rightarrow u \ge v)$, equivalently, $(u \not\le v$ or $u \ge v)$, that is, $u \ge v$ or $u_i > v_i$ for some $i$. The usual definition of $u$ being a *maximal* in a set $V$ is $u \not\prec v$ for all $v$ from $V$. This coincides with $u$ being *Pareto-optimal*, better known by its definition $u \prec v$ for no $v$ in $V$. Also, $u \prec v$ iff ($\mathbb{1}u < \mathbb{1}v$ and $u \le v$), as is easily verified.

## G.2 Axioms

There are $n$ *agents*[9] $N := \{1, \dots, n\}$. Unless stated otherwise, $i$ is in $N$. Let there be a number $E$, the *endowment* (or estate). Let there be a parameter $\pi$ from some multi-dimensional set, including any preferences of the agents. Let $\Xi_\pi$ be a subset of $\mathbb{R}^n_{\ge 0}$. The *allocation problem* is to determine, for all $i$ in $N$, a number $x_i(\pi)$, the *award* (or allotment, allocation, and so on), where $x(\pi) \in \Xi_\pi$, such that there is *sub-balance*,

$$\mathbb{1}x(\pi) \le E \tag{6}$$

that is, to distribute $E$ in portions $x_i(\pi)$ to $i$. So, the set of profiles $x$ is implicitly defined by conditions imposed on $x$, as expressed by $\Xi_\pi$. For example, $\pi = E$ and $\Xi_i(\pi) = [0, E]^n$, that is, $0 \le x_i(\pi) \le E$ for all $i$ in $N$. If $\mathbb{1}x(\pi) = E$ then there is *balance*.[10] Any remainder $E - \mathbb{1}x(\pi)$ is wasted. The parameter $\pi$ and conditions $\Xi_\pi$ imposed on $x(\pi)$ are as follows.

The *maximum allocation problem* or *bank dissolution problem*[11] is the allocation problem with joint awards boundedness where claimant jointly wish to maximise their award:

$$x(\pi) \in \max \Xi_\pi \tag{7}$$

The max is with respect to the joint $\le$, that is, $y \ge x(\pi) \Rightarrow y \le x(\pi)$ for all $y$. Therefore, this condition is equivalent to $x(\pi) \not\prec y$ for all $y$

[9] These agents may only be the employees of an employer in the main text and not the whole group.

[10] As in [39, p.42] but also 'efficiency' [38, p.252].

[11] The reference to a bank is only to conform to the usual term 'bankruptcy'. Various distinctions between bankruptcy and rationing are possible: bankruptcy could deal only with $\mathbb{1}c > E$ while rationing with both $\mathbb{1}c > E$ and $\mathbb{1}c \le E$ but in times of abundance, the term 'rationing' is not applicable; or rationing could be characterised by single-peaked preferences where the peak may be between $c_i$ and $E$; or rationing could consider private (subjective) preferences whereas bankruptcy would deal with public (objective) claims.

in $\Xi_\pi$ and thus Pareto optimal, as set out at the end of Section G.1. For example, $x_i$ is in $[0, E]$ and each preference relation is single-peaked at $E$.

**Proposition 7** *The award $x$ is* endowment-monotonous*, that is, $E' \geq E$ implies $x'_i \geq x_i$ for all $i$ in $N$.*

**Proof**. Equation 6 ($\mathbb{1}x \leq E$) lets every increase of $E$ allow further maximisation of $x$, Equation 7. ■

For all $i$ let there be a number $c_i$, called *claim* of $i$, who is now called *claimant*.[12] The *claims problem*[13] or *bankruptcy problem* is the maximum allocation problem where $\mathbb{1}c > E$ and no solution is offered for $\mathbb{1}c \leq E$. The case $\mathbb{1}c \leq E$ is called *bank solvency* though that term does not express that the bank redeems all its debts.[14] From now on, only mention $c_i$ or the entire $c$ as the argument of $x_i$.

**Proposition 8** *If $\mathbb{1}c > E$ then $\mathbb{1}x(c) < \mathbb{1}c$, so-called* joint awards boundedness. *In that case, $x_i(c_i) < c_i$ for some $i$.*

**Proof**. Supposing $\mathbb{1}x \geq \mathbb{1}c$ would contradict Equation 6. Were $x \geq c$ then $\mathbb{1}x \geq \mathbb{1}c$. ■

The $c_i$ is a *bounding claim* if $x_i(c_i) \leq c_i$. If so for everyone,

$$x(c) \leq c \tag{8}$$

then $x$ obeys *awards boundesness.*[15] So, if there is more than enough for everyone, then still nobody's award will be strictly greater than his or her claim and the surplus is discarded.[16]

Let there be numbers $\mu_i$, the *claims guarantee*, *risk bound*, or *soft bound*[17] (bound of claims, that is) such that $x_i|_{[0,\mu_i]} = \mathrm{id}$. If so for all,

$$c_i \leq \mu_i \text{ implies } x_i(c_i) = c_i \text{ for all } i \text{ in } N \tag{9}$$

then $\mu$ is the profile called correspondingly. Imposing Equation 9 is called *exemption* if $x(c) \leq c$ (Equation 8) because agents having low claims 'cannot be held responsible for the shortage' and thus should receive their claim in full [19, p. 317]. The term 'risk bound' for $\mu_i$ is explained by the following, which does not depend on $x(c) \leq c$, Equation 8.

[12] The term 'claim' is not common for the peaks of single-peaked preferences which are not necessarily at the highest possible value. For the problem where $\mathbb{1}c < E$ the term 'claim' is maintained [38, p. 252, 291ff].

[13] In [39, p. 42] the definition extends to $\mathbb{1}c = E$ but this case is no real problem and would be inconvenient in the present context.

[14] Compare to the festive redemption of all debts [26, p. 397]. It would be even more festive if awards were greater than the claims, as for one-sidedness.

[15] Conventionally, 'claims boundedness' [39, p.42] but the claims are not bounded.

[16] The allocation problem is *one-sided* [32, p. 599] if $x \leq c$ for $\mathbb{1}c \geq E$ and $x \geq c$ for $\mathbb{1}c \leq E$. For example, each claimant $i$ has a preference relation on the possible awards $x_i$ that has a single peak at $c_i$. Single-sidedness may be dropped [38, pp. 291ff].

[17] For sequential allotment, synonyms are 'reference allocations', 'reference points', 'collection of shares', and 'guaranteed levels', but they are not necessarily the greatest guarantees as they change during the computation [7, pp.641-642]. For the rationing problem, a similar if not identical concept is called benchmark allocation, $\omega$-guarantee, and initial endowment [32, pp. 588, 593].

**Proposition 9** *If $\mathbb{1}c > E$ then $\mu_i < c_i$ and $x_i(c_i) < c_i$ for some $i$.*

**Proof**. Proposition 8: Joint awards boundedness, $\mathbb{1}x < \mathbb{1}c$, tells that $x_i < c_i$ for some $i$. Equation 9: $\mu_i < c_i$. ■

The term 'soft bound' stresses that $j$ other than $i$ in Proposition 9 may have $\mu_j < c_j$ yet $x_j(c_j) \geq c_j$.

Consider the case $\mathbb{1}\mu > E$. It may happen that $c = \mu$ and $\mathbb{1}c > E$. Then $x(c) = x(\mu) = \mu = c$ so $\mathbb{1}x(c) = \mathbb{1}c > E$, contradicting Equation 6, $\mathbb{1}x(c) \leq E$, and there would be no solution.[18] Therefore, impose

$$\mathbb{1}\mu \leq E \tag{10}$$

which is called the *claims guarantee condition.*

The $\mu_i$ is called a *non-diminishment bound* if

$$c_i > \mu_i \text{ implies } x_i(c_i) \geq \mu_i \tag{11}$$

for all $i$ in $N$. In words, claims above the claims guarantee yield an award also above the claims guarantee. This condition should not be added to the axioms if the award if required to be monotonous. This is obvious from a graph of $x$ but formalised as follows.

**Proposition 10** *If $x_i$ is weakly monotonous (non-decreasing as a function of $c_i$) then $\mu_i$ is a non-diminishment bound, Equation 11.*

**Proof**. Suppose $c_i > \mu_i$ but $x_i(c_i) < \mu_i$. Choose a claim $d_i$ such that $x_i(c_i) < d_i \leq \mu_i$, for example $d_i = \mu_i$. From $d_i \leq \mu_i$ follows not only $d_i < c_i$ but also $x_i(d_i) = d_i$ because of exemption, Equation 9. So $x_i(d_i) > x_i(c_i)$, contradicting monotonicity. ■

Continuity is only partially guaranteed, as the following proposition and example show.

**Proposition 11** *If $\mu_i$ is a* non-diminishment bound *(Equation 11) then the award $x_i$ is right-continuous in $\mu_i$.*

**Proof**. Let $\epsilon$ in $(0, \infty)$ and choose $\delta$ in $(0, \epsilon]$. Let $\gamma$ in $(\mu_i, \mu_i + \delta)$. The definition of continuity requires $x_i(\gamma)$ in $(\mu_i - \epsilon, \mu_i + \epsilon)$. Non-diminishment bound: $x_i(\gamma) \geq \mu_i > \mu_i - \epsilon$. Exemption, Equation 8: $x_i(\gamma) \leq \gamma < \mu_i + \delta \leq \mu_i + \epsilon$. ■

The proof can also apply the 'squeeze theorem' by squeezing $x_i$ between the diagonal and the constant function.

**Example 11** *If the non-identical part of $x_i$ is a 'squeezed Dirichlet function,'*

$$x_i|_{[\mu_i,\infty)}(\gamma) = \begin{cases} \gamma & \text{if } \gamma \text{ is rational (a fraction)} \\ \mu_i & \text{otherwise} \end{cases}$$

*then $x_i|_{[\mu,\infty)}$ is nowhere continuous except in $\mu_i$.*

Such pathological functions are excluded by imposing continuity.

[18] This is not resolved by imposing that $\mu$ be a non-diminishing bound profile, Equation 11, which means that $c \geq \mu$ implies $x(c) \geq \mu$, so $\mathbb{1}x(c) \geq \mathbb{1}\mu$. For, it may turn out that $c \geq \mu$ and $\mathbb{1}\mu > E$ (so, the case $\mathbb{1}c > E$). Again, in view of Equation 6, no solution would be possible.

## G.3 Definition

The displayed equations (or axioms) define the *maximum allocation problem with soft bounds*, which is the topic of the sequel. As set out above, Equation 11 can be omitted if monotonicity is imposed. These axioms, except maximality, Equation 7, determine the region $\Xi_\pi$ for $\pi = (E, c, \mu)$, which specifies $n$. Judging from overviews by Thompson [38, 39] the maximum allocation problem with soft bounds has not been considered in the literature. Thompson remarks 'Imposing this bound [$c_{\max}$, on the claims] restricts somewhat the scope of the rule but it permits a very simple (piecewise linear) representation' [38, p.259, n.11] but that is a representation of the Talmud rule [12, p.284].

**Proposition 12** *The following holds irrespective of whether $\mu$ is a non-diminishment bound profile, Equation 11.*

1. identity solution*: $\mathbb{1}c \leq E$ is equivalent to $x = id$, that is, $x_i(c_i) = c_i$ for all $i$.*
2. *If $\mathbb{1}c < E$ then $\mathbb{1}x < E$; if $\mathbb{1}c \geq E$ then $\mathbb{1}x = E$.*
3. *If $\mathbb{1}c \geq E$ then $c \not< \mu$.*
4. *If $\mathbb{1}c \geq E$ then $c \not\prec \mu$.*

*The last two statements have been explained at the end of Section G.1.*

**Proof**. Item 1: From $x = c$ follows $E \geq \mathbb{1}x = \mathbb{1}c$ because of Equation 6. Conversely, let $\mathbb{1}c \leq E$. The following shows that $x = c$ satisfies all conditions on $x$. Equation 6: $\mathbb{1}x = \mathbb{1}c \leq E$. Equation 8: $\mathbb{1}x \leq \mathbb{1}c$, so Equation 7 (maximality) guarantees uniqueness of $x$. Equation 9 (exemption) and Equation 11 (if imposed) are trivially true.

Item 2: Consider only $\mathbb{1}c > E$ as the rest follows from Item 1. So $x \neq c$. Let $i$ be such that $x_i < c_i$. (Equation 8: $x \leq c$.) Equation 9: $c_i > \mu_i$. Assume $\mathbb{1}x < E$ to derive a contradiction. Let $\epsilon := \min\{c_i - x_i, E - \mathbb{1}x\}$ and define

$$y_j := \begin{cases} x_i + \epsilon & \text{if } j = i \\ x_j & \text{otherwise} \end{cases}$$

for $j$ in $N$. Then $y$ satisfies the following conditions. Equation 6: $\mathbb{1}y = \mathbb{1}x + \epsilon \leq E$. Equation 8: $\mathbb{1}y = \mathbb{1}x|_{N\setminus\{i\}} + x_i + \epsilon \leq \mathbb{1}c$. Equation 9 does not apply because $c_i > \mu_i$. If Equation 11 is imposed, then $x_i \geq \mu_i$ so it also holds for $y$ because $y_i > x_i$. As $y$ is in $\Xi_\pi$, the region defined by these conditions, $x$ violates maximality, Equation 7.

Item 3: Suppose the contrary, $c < \mu$. Then $E \leq \mathbb{1}c < \mathbb{1}\mu$, which contradicts Equation 10.

Item 4: Suppose the contrary, $c \prec \mu$. Let $j$ be such that $c_j < \mu_j$. Then $\mathbb{1}c = c_j + \mathbb{1}c|_{N\setminus\{j\}} < \mu_j + \mathbb{1}c|_{N\setminus\{j\}} \leq \mathbb{1}\mu \leq E$, a contradiction with $\mathbb{1}c \geq E$. ■

The following example of the uniform gains illustrates how $\mu_i$ is an initial endowment, but more such endowment profiles are defined in the course of the computation for the uniform rule.

**Example 12** *Consider the uniform rule for $\mathbb{1}c > E$ (the uniform gains) as defined by $x_i = \min\{\lambda, c_i\}$ where $\lambda$ solves $\mathbb{1}x = E$. Then $\mu_i = E/n$ is* *a constant claims guarantee and a non-diminishment bound. This is seen as follows. First of all, $\lambda \geq E/n$ for otherwise, $x_i \leq \lambda < E/n$ so $\mathbb{1}x < E$, a contradiction. If $c_i \leq E/n$ then $c_i \leq \lambda$ so $x_i = c_i$ and $\mu_i$ is a claims guarantee. If $c_i > E/n$ then there are two cases. One case is $c_i \leq \lambda$ so $x_i = c_i > E/n$ The other case is $c_i > \lambda$, which implies $x_i = \lambda \geq E/n$. In both cases, $x_i \geq E/n$ so $\mu_i$ is a non-diminisment bound.*

An example of $\mu$ having a meaning that does not depend on the computation is an allocation problem where $\mathbb{1}c > E$, the $\mu_i$ are individual endowments (in the sense of possessions) and $c_i$ are the unique peaks of a preference; this is the case in [41, p. 793] where claimants whose claims can not be honoured in full, receive an award proportional to the individual endowment, not (partially) proportional to the claim, as will be required below.

It may be the case that $\mu$ is public but $E$ is private, as in the following.

**Example 13 (Dutch banks)** *The Dutch central bank guarantees that customers $i$ from most Dutch banks receive their claims up to a claims guarantee $\mu_i = \mu_1 = 10^5$ euro from their bank when it goes broke [33]. Suppose all have $c_i \geq 10^5$ euro on their account of the Dutch XYZ bank: $c \geq \mu$. Everyone should receive at least the non-diminishment bound: $x_i \geq \mu_i$. Therefore, $\mathbb{1}x \geq \mathbb{1}\mu$. There are $n = 10^4$ customers so $\mathbb{1}x \geq \mathbb{1}\mu = 10^9$. The bank does not own a billion euros: $E < \mathbb{1}\mu$. It should never have promised these claims guarantees: $\mathbb{1}x > E$. So there is bankruptcy because of $\mathbb{1}\mu > E$ rather than $\mathbb{1}c > E$. Fortunately, the central bank will come to the rescue.*

The following example illustrates discarding $\mathbb{1}c - E$.

**Example 14** *Two agents estimated their labour duration as $\mu = (2, 6)$ hours. Their total estimated labour duration is $E = \mathbb{1}\mu = 8$. They worked $c = (4, 3)$ so $\mathbb{1}c < E$. So $x = c$, that is, the accounted labour duration $x_1 = 4$ of agent 1 is 2 hours more than the underestimate $\mu_1 = 2$ and $x_2 = 3$ is 3 hours less than the overestimate $\mu_2 = 6$. This surplus estimate $\mu_2 - c_2 = 6 - 3 = 3$ went to agent 1 but the sum $\mu_1 + 3 = 5$ is capped to $c_1 = 4$ because nobody should be awarded more than the actual labour duration. Therefore, $E - \mathbb{1}x = 1$ is an unused surplus estimate.*

## G.4 Solution: Preparation

The following is a first acquaintance with the problem of finding a formula.

**Example 15** *The formula $x_i = \min\{c_i, \mu_i\}$ is not a solution. For, assume $c_j < \mu_j$ for a unique claimant $j$. Then $\mathbb{1}x = c_j + \mathbb{1}\mu|_{N\setminus\{j\}}$ would hold, so $\mathbb{1}x < \mathbb{1}\mu \leq E$, that is, not all of $E$ is distributed, which is not Pareto optimal.*

As in [41, p. 793], distinguish between *demanders* $D := \{i \in N \mid c_i > \mu_i\}$ and *suppliers* $S := \{i \in N \mid c_i < \mu_i\}$. The non-suppliers are $S^\neg = \{i \in N \mid c_i \geq \mu_i\}$. The non-demanders are $D^\neg = \{i \in N \mid c_i \leq \mu_i\}$ and they receive $x_i(c_i) = c_i$. The remainder $E - \mathbb{1}c|_{D^\neg}$ is distributed over the demanders. provided $D \neq \emptyset$, that is, $c \not\leq \mu$. The claims guarantees ensure that the remainder is non-negative: $E - \mathbb{1}c|_{D^\neg} \geq E - \mathbb{1}\mu|_{D^\neg} \geq E - \mathbb{1}\mu \geq 0$

even if $D^{\neg} = \emptyset$. This remainder consists of endowment in excess to the total claims guarantee, $E - \mathbb{1}\mu$, the total surplus $\mathbb{1}(\mu - c)|_S = \mathbb{1}(\mu - c)|_{D^{\neg}}$, and $\mathbb{1}\mu|_D$, as demanders receive at least the non-diminishment bound $\mu_i$.

Introduce four desirable properties.[19] *Claims monotonicity*: $c_i \geq d_i$ implies $x_i(c_i) \geq x_i(d_i)$ for all $i$ in $N$. (Nobody wants to invest more but be awarded less.) *Claims continuity*: $x_i$ is a continuous function of $c_i$ for all $i$ in $N$. (It would be unfair or odd if an infinitesimal increase of the claim would yield a considerably larger award.) *Others-oriented claims monotonicity* [38, p. 45] or rather, antitonicity: $c_i \geq c_i'$ implies $x_j \leq x_j'$ for all $j$ in $N \setminus \{i\}$. *Others-oriented claims continuity*: $x_i$ is a continuous function of $c_j$ for all $i$ in $N$ and $j$ in $N \setminus \{i\}$. (If $j$ gradually raises $c_j$ then a drop in $x_i$ should be predictable.)

The outcome need not be strategy-proof because the claims are considered objective quantities.

## G.5 Fixed-Path Rationing for Soft Bounds

The award is partially proportional to the claim when applying the uniform gains method, as follows. For $i$ in $D$ let $\xi_i(\lambda) := \min\{\lambda c_i, c_i - \mu_i\}$ for some $\lambda$ and determine $\lambda$ from $\mathbb{1}\xi(\lambda) = E - \mathbb{1}c|_{D^{\neg}} - \mathbb{1}\mu|_D$. (This is a weighted gains having weights $c_i$ in $\lambda c_i$ which are considered exogenous.) Add $\xi_i(\lambda)$ to the minimum award $\mu_i$. So, the remainder is distributed proportional to $c_i$ but the award will not exceed $c_i$. Notably, $E - \mathbb{1}c|_{D^{\neg}} - \mathbb{1}\mu|_D \geq E - \mathbb{1}\mu|_{D^{\neg}} - \mathbb{1}\mu|_D = E - \mathbb{1}\mu \geq 0$ also if $D^{\neg} = \emptyset$. Inserting the condition $\mathbb{1}c|_{D^{\neg}} > E$ avoids superfluous computation of the trivial solution:

$$x_i(c_i) = \begin{cases} \mu_i + \xi_i(\lambda) & \text{if } i \in D \text{ and } \mathbb{1}c|_{D^{\neg}} > E \\ c_i & \text{otherwise} \end{cases}$$

So, $\lambda$ is a solution of $\sum_{i \in D} \min\{\mu_i + \lambda c_i, c_i\} = E - \mathbb{1}c|_{D^{\neg}}$. This sum naturally extends to $i$ in $D^{\neg}$, so let

$$\chi_i(\lambda) := \min\{\mu_i + \psi_i(\lambda), c_i\} \tag{12}$$

where $\lambda c_i$ has been generalised to $\psi_i(\lambda)$ for (non-negative) monotonous functions $\psi_i$ obeying $\psi_i(0) = 0$ for $i$ in $D$. Let $\mathbb{1}\chi(\lambda) = E$ according to Item 2 in Proposition 12. This yields $\lambda$ provided $D \neq \emptyset$ and $\mathbb{1}c \geq E$. (Otherwise, there is no such $\lambda$.) This is fixed-path rationing with the $N$-path $\mu_i + \psi_i(\lambda)$ considered exogenous. Let

$$x_i(c_i) = \begin{cases} \chi_i(\lambda) & \text{if } \mathbb{1}c > E \\ c_i & \text{otherwise} \end{cases} \tag{13}$$

be *fixed-path rationing for soft bounds*. The condition $D \neq \emptyset$ need not be added to $\mathbb{1}c > E$ in Equation 13 because if $c_i \leq \mu_i$ then the minimum is $x_i(c_i) = c_i$.

**Example 16** *Let* $E := 5\frac{2}{3}$ *and* $c := (1, 2, 3)$ *for* $\mu := (2, 1, 1)$. *Equation 13, yields* $\lambda = \frac{5}{9}$ *and* $x = \left(1, 2, \frac{8}{3}\right)$. *So, the award of agent 2 surpasses the claims guarantee but the award still equals the claim.*

[19] Monotonicity is in [39, p.45] and continuity is defined similarly.

It is impossible to replace $\mu_i + \psi_i(\lambda)$ (for example, $\mu_i + \lambda c_i$) in Equation 12 with $\lambda c_i$ and increase proportionality of $x_i(c_i)$ to $c_i$, that is, have $x_i(c_i) = \lambda c_i$ for $i$ in $D$. For, $\lambda \leq 1$ (awards boundedness, Equation 8). Consider $\mathbb{1}c > E$. The case $\lambda = 1$ is excluded because it would imply $x(c) = c$ and thus $\mathbb{1}x(c) > E$, a contradiction. So $\lambda < 1$. But then $x_i(\mu_i) = \lambda\mu_i < \mu_i$, contradicting Equation 9.

**Proposition 13** *The fixed-path rationing for soft bounds, Equation 13, solves the maximum allocation problem with soft bounds, is continuous and strongly monotonous, continuous and weakly monotonous with respect to others, and resource-monotonous.*

**Proof**. Equation 6 if $\mathbb{1}c > E$ then $\mathbb{1}x = E$ by definition of $x = \chi$ and trivially if $\mathbb{1}c \leq E$. Equation 8, Equation 9, and (if imposed) Equation 11 because of min. Equation 7: $x$ can not be increased because $\mathbb{1}x = E$ or $x = c$. Continuity: the min is continuous in $c_i$. Right-continuity in $\mu_i$ was proven in Proposition 11. Strong monotonicity: $c_i$ occurs in both arguments of min. Continuity and strong monotonicity in $c_i$ with respect in $c_j$ for $j \neq i$: $\chi_i = E - \mathbb{1}\chi|_{N\setminus\{i,j\}} - \chi_j$ if $\mathbb{1}c > E$. However, constancy if $\mathbb{1}c \leq E$. Endowment-monotonicity: if $E$ increases, $\lambda$ will not strictly decrease, so no $\chi_i(\lambda)$ will either. ■

Endowment-monotonicity was alread proven in Proposition 7.

## G.6 Maximum Allocation for Reduction Boundaries

Example 16 illustrated how the case $c_i > \mu_i$ may still yield $x_i(c_i) = c_i$. So, $x_i = \mathrm{id}$ beyond the soft bound $\mu_i$. This is undesirable if a simple closed-form expression is sought (one without iterations and superfluous if-statements) or agents $i$ claiming more than $\mu_i$ should be 'punished' by receiving less than their claim. The following elaborates on these requirements.

Item 1 in Proposition 12 indicates that two cases should be distinguished. One case is $\mathbb{1}c \leq E$, for which the award equals the claim: $x = c$. The other case is bankruptcy: $\mathbb{1}c > E$, which is considered next. Let $\omega_i$ be a number depending on the variables and typically obeying $\omega_i \leq \mu_i$. Equation 9 is identical to

$$x_i(c_i) := \begin{cases} \phi_i(c_i) & \text{if } c_i > \omega_i \\ c_i & \text{if } c_i \leq \omega_i \end{cases} \tag{14}$$

where $\phi_i$ is some function of $c_i$ for $c_i \geq \omega_i$ (and parametrised by other variables). The case $\omega_i > \mu_i$ is also allowed, but then $\omega_i$ could replace $\mu_i$. The obvious choice $\omega = \mu$ is justified as follows.

Suppose

$$\phi_i|_{(\omega_i,\zeta)} \neq \mathrm{id} \text{ for all } \zeta \text{ in } (\omega_i, \infty) \tag{15}$$

which is no loss of generality: For, if there were a $\zeta$ for which equality held, then $\phi_i|_{(\zeta,\infty)}$ and $\zeta$ could replace $\phi_i$ and $\omega_i$, respectively.

If $\mu_i$ is the largest number such that $x_i|_{[0,\mu_i]} = \text{id}$ (see just before Equation 9) then $\mu_i$ is called *exemption boundary* and if so for everyone,

$$\mu_i = \max\{\nu \mid x_i|_{[0,\nu]} = \text{id}\} \text{ for all } i \text{ in } N \tag{16}$$

then $\mu$ is an *exemption boundary profile.* If also $x(c) \leq c$ (Equation 8) then $\mu_i$ is called *reduction boundary* and $\mu$ the corresponding profile. Equivalently, $\mu_i = \inf\{\zeta \in (\mu_i, \Gamma] \mid x_i(\zeta) < c_i\}$ where $\Gamma$ is the largest claim for which $x_i(\Gamma)$ is defined. So, $\mu$ determines aforehand, irrespective of the claims $c_i$, that the awards drop if the claim slightly surpasses $\mu_i$. For bankruptcy, this would be unfair if there is enough to satisfy claims (slightly) beyond the claims guarantee $\mu_i$. However, if the endowment $E$ is not exogenous, then imposing reduction boundaries may deter agents from overreporting $E$, such as when $\mu_i$ is an estimated labour duration and $E = \mathbb{1}\mu$.

**Proposition 14** *In conclusion,* $\omega = \mu$.

**Proof**. If $\omega_i < \mu_i$ were the case then Exemption, Equation 9, would imply $\phi_i|_{(\omega_i,\mu_i)} = \text{id}$, a contradiction with Equation 15. Were $\omega_i > \mu_i$ then the definition of $x_i$ having bounds $\omega_i$, Equation 14, would yield $x_i|_{(\mu_i,\omega_i)} = \text{id}$, contradicting Equation 16. ■

Add the case $\mathbb{1}c \leq E$ again:

$$\begin{array}{ll} \text{if } \mathbb{1}c > E \text{ then} & x_i(c_i) := \begin{cases} \phi_i(c_i) & \text{if } c_i > \mu_i \\ c_i & \text{if } c_i \leq \mu_i \end{cases} \\ \text{if } \mathbb{1}c \leq E \text{ then} & x_i(c_i) := c_i \end{array}$$

The formule for $\mathbb{1}c > E$ can not be used for $\mathbb{1}c \leq E$. Suppose it would. Consider $c = (1, c_2)$ for $2 < c_2 \leq 5$ and $E = 6$ so $x = c$. Let $\mu = (4, 2)$ so $c_2 > \mu_2$ and $\phi_2(c_2) = c_2$ for all $c_2$. So $\phi|_{(\mu_2,\zeta)} = \text{id}$ for $\mu_2 = 2$ and $\zeta = 5$, contradicting Equation 15. So, $x = c$ for $\mathbb{1}c \leq E$ needs to be mentioned separately. The two cases are combined as follows:

$$x_i(c_i) := \begin{cases} \phi_i(c_i) & \text{if } c_i > \mu_i \text{ and } \mathbb{1}c > E \\ c_i & \text{otherwise} \end{cases} \tag{17}$$

Consider $\mathbb{1}c > E$ again. As in Section G.5, the remainder after serving the suppliers, $R := E - \mathbb{1}c|_{D^\neg} - \mathbb{1}\mu|_D$, is to be distributed over the demanders $i$ in $D$ in addition to their minimum award $\mu_i$. Suppose $R$ is distributed proportionally to not just $c_i$ but more generally, to $c_i - \nu_i$ for some $\nu_i$. So $\phi_i(c_i) = \mu_i + (c_i - \nu_i)R/d$ where $d$ is a scaling. More generally still, suppose $\phi_i(c_i) = \mu_i + \rho_i(c_i - \nu_i)R/d$, where $\rho_i$ for $i$ in $D$ are (non-negative) monotonously increasing functions obeying

$$\textstyle\sum_{i\in D} \rho_i(z_i) = \sum_{i\in D} z_i \text{ for all } n\text{-vectors } z \tag{18}$$

that is, $\mathbb{1}\rho(z) = \mathbb{1}z$ and in particular, $\rho_i(0) = 0$. The scaling is $d = \mathbb{1}\rho(c - \nu) = \sum_{i\in D} \rho_i(c_i - \nu_i)$, as follows from $\mathbb{1}\phi = \mathbb{1}\mu|_D + R$. So, $\mathbb{1}\rho(c - \nu) = \mathbb{1}(c - \nu)|_D$. As $\phi_i(\lambda) \downarrow \mu_i$ if $\lambda \downarrow \mu_i$ for all $i$, Proposition 11, the only possibility is $\nu_i = \mu_i$. So, let

$$\alpha := \frac{\mathbb{1}c - E}{\mathbb{1}(c - \mu)|_D} \quad \text{and} \quad \beta := \frac{E - \mathbb{1}c|_{D^\neg} - \mathbb{1}\mu|_D}{\mathbb{1}(c - \mu)|_D} \tag{19}$$

then (for $i$ in $D$)

$$\begin{aligned} \phi_i(c_i) &= c_i + \alpha(\mu_i - c_i) \\ &= \mu_i + \beta(c_i - \mu_i) \end{aligned} \tag{20}$$

as $\alpha + \beta = 1$. This also follows from positing that $\phi_i(c_i) = \alpha\mu_i + \beta c_i$ combined with $\phi_i(\mu_i) = \mu_i$ and $\mathbb{1}x(c) = E$. Equation 17 now defines *maximum allocation for reduction boundaries.* Note that $\alpha$ (as well as $\phi$ and $\rho$) are only defined if $\mathbb{1}c > E$.

**Proposition 15** *The* $\alpha$ *from Equation 19 obeys* $0 < \alpha \leq 1$. *Formula 17 is well-defined and solves the maximum allocation problem with soft bounds.*

**Proof**. If $\mathbb{1}c > E$ then $D \neq \emptyset$ because $D = \emptyset$, which is equivalent to $c \leq \mu$, would imply $\mathbb{1}c \leq E$. So, the divisor $\mathbb{1}(c-\mu)|_D > 0$. The numerator $\mathbb{1}c - E > 0$ so $\alpha > 0$. Also $\mathbb{1}c - E \leq \mathbb{1}(c-\mu) = \mathbb{1}(c-\mu)|_D + \mathbb{1}(c-\mu)|_{D^\neg}$ and the latter term is $\leq 0$ by definition. So $\alpha \leq 1$.

The formula meets all conditions for being a solution, as follows. Sub-balance, Equation 6: $\mathbb{1}x = E$ by construction for $\mathbb{1}c > E$. If $\mathbb{1}c \leq E$ then $\mathbb{1}x = \mathbb{1}c$. Bounding claims, Equation 8: $x_i = \alpha(\mu_i - c_i) + c_i \leq c_i$ for $i$ in $D$ and $x_i = c_i$ elsewhere. Exemption, Equation 9: by design. Risk boundary, Equation 16: would generally $\phi_i = c_i$ then $\alpha\mu_i + (1-\alpha)c_i = c_i$ but $\alpha > 0$. Non diminishment bound: Equation 11: $\phi_i(\mu_i) = \mu_i$ and $1-\alpha \geq 1$ so $\phi_i$ is non-decreasing in $c_i$. Maximality, Equation 7: if $\mathbb{1}c > E$ then $\mathbb{1}x = E$ and if $\mathbb{1} \leq E$ then $x = c$ so $x$ cannot increase. ∎

Equation 20 proves monotonicity of $x_i$ in $c_i$. Proposition 10 and Proposition 11 suffice for proving continuity.

**Proposition 16** *The award* $x_i(c_i)$ *from Equation 17 is antitone and continuous with respect to* $c_j$ *for* $j \neq i$.

**Proof**. Define $\Phi_i(c_j) = \phi_i(c_i)$ and $X_i(c_j) = x_i(c_i)$ to stress the dependence of $\phi_i$ and $x_i$ on $c_j$. Let $\Gamma_j := \mathbb{1}(c-\mu)|_{D\setminus\{j\}} - \mu_j$ and $\Delta_j := E - \mathbb{1}c|_{N\setminus\{j\}}$, which are independent of $c_j$. (As before, $D$ are the demanders, the $i$ for whom $c_i > \mu_i$.) Equation 17 is

$$X_i(c_j) = \begin{cases} \Phi_i(c_j) & \text{if } c_i > \mu_i \text{ and } c_j > \Delta_j \\ c_i & \text{otherwise} \end{cases}$$

where $\Phi_i(c_j) = \alpha\mu_i + \beta c_i$ for

$$\alpha = \frac{c_j - \Delta_j}{\mathbb{1}(c-\mu)|_D} = A(c_j) := \frac{c_j - \Delta_j}{\Gamma_j + c_j}$$

and $\beta = 1 - \alpha$. If $c_i \leq \mu_i$ or $c_j \leq \Delta_j$ then $X_i(c_i) = c_i$, which is a constant, so continuous and antitone. Consider $c_i > \mu_i$ and $c_j > \Delta_j$. Then $D \neq \emptyset$ so $\mathbb{1}(c-\mu)|_D \neq 0$ and $\alpha$ is defined. If $c_j \downarrow \Delta_j$ then $\alpha \downarrow 0$, hence $\Phi_i(c_j) \to c_i$ (actually, $\Phi_i(c_j) \uparrow c_i$). So $X_j = \Phi_j$ is continuous in $c_j$, in particular, right-continuous at $\Delta_j$. Finally, for any $d$ such that $d > c_j$, a little calculus shows that $A(d) > A(c_j)$, where $A(c_j)$ is $\alpha$ as a function of $c_j$, as displayed. In $X_i(c_j) = \Phi_i(c_j) = (\mu_i - c_i)A(c_j) + c_i$ the factors are $\mu_i - c_i < 0$ by assumption and $A(c_j) > 0$ so $X_i(d) < X_i(c_j)$, that is, $X_i$ is antitone in $c_j$. ∎

The case $c_i > \mu_i$ and $\mathbb{1}c > E$ also confirms endowment monotonicity: $x_i = \alpha(\mu_i - c_i) + c_i$ where $\alpha > 0$. Suppose $E$ increases. Then $\alpha$ as in Equation 19 decreases so $x_i$ increases. If $\mathbb{1}c > E$ switches to $\mathbb{1}c \leq E$ then fewer cases of $x_i(c_i) = \phi_i(c_i)$ occur, where $\phi_i(c_i) \leq c_i$. So $x_i$ does not decrease.

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